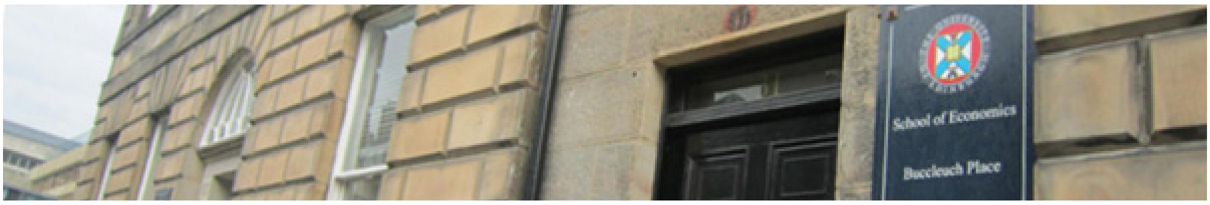
Matching and Sorting in a Global Economy



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Matching and Sorting in a Global Economy*

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Abstract

We develop a neoclassical trade model with heterogeneous factors of production. We consider a world with two factors, labor and "managers", each with a distribution of ability levels. Production combines a manager of some type with a group of workers. The output of a unit depends on the types of the two factors, with complementarity between them, while exhibiting diminishing returns to the number of workers. We examine the sorting of factors to sectors and the matching of factors within sectors, and we use the model to study the determinants of the trade pattern and the effects of trade on the wage and salary distributions. Finally, we extend the model to include search frictions and consider the distribution of employment rates.

Keywords: heterogeneous labor, matching, sorting, productivity, wage distribution, international trade.

JEL Classification: F11, F16

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1 Introduction

In this paper, we study how international trade affects the sorting of heterogenous workers and managers into industries and the matching of workers with managers in production units. It is by now well known that firms in the same industry differ in size, in the compositions of their workforces, in the technologies and capital goods they use, and in the wages they pay to their workers. Industries differ in factor intensities and in the marginal contributions of worker and managerial ability to firm productivity. Workers differ in physical attributes, in cognitive abilities, and in their education, training, and experience. Although some studies of international trade have examined the assignment of heterogeneous labor to different sectors and others have considered the matching of workers to heterogeneous teammates or technologies, relatively little is known about the general problem of how factors sort and match in the open economy when several of these factors are differentiated, when fixed quantities of one impart decreasing returns to the others, and when industries differ in their factor intensities and in the usefulness of factor "quality." Our paper addresses these more general, allocational issues and their implications for factor rewards. Because workers and managers are heterogeneous, our analysis sheds light on the impact of trade on the *distribution* of wages and managerial salaries, and thereby on the impact of trade on earnings inequality.

By allowing for worker, manager, and industry heterogeneity, we can better understand a number of issues concerning the pattern and consequences of international trade. First, we can study how countries' *distributions* of differentiated factors, in conjunction with their aggregate endowments of these factors, determine their comparative advantage in the various sectors. Bombardini et al. (2012) provide evidence, for example, that countries' skill dispersions have a quantitatively similar impact on trade flows as do their aggregate endowments of human capital. Second, we can investigate how trade influences factor returns across the entire income distribution, affecting more than just the relative compensation paid to one factor versus another or to workers employed in one industry versus another. These additional dimensions of inequality can be useful for understanding recent findings of substantial variation in wages that is not easily explained by observable worker characteristics. Helpman et al. (2012) show, for example, that within-industry wage variation accounts for a majority of wage inequality in Brazil even after controlling for workers' occupations. Third, we can examine how globalization affects the distribution of employment rates across skill levels in a setting with search-and-matching frictions.

The literature on the *sorting* of workers to industries includes recent work by Costinot (2009), Costinot and Vogel (2010), and Ohnsorge and Trefler (2007), as well as earlier work by Mussa (1982) and Ruffin (1988).[1] All of these authors emphasize the comparative advantage that the various types of labor have when employed in different industries. They study the determinants of

[1] We use the term "sorting" to refer to the allocation of heterogeneous factors to different industries and the term "matching" to refer to the combination of differentiated factors within an industry.

the trade pattern in countries that differ in the compositions of their labor forces and the impact that trade has on income inequality across the skill or ability spectrum. But most assume a linear relationship between labor input (of a given quality) and output or, what amounts to the same, an absence of interactions between quantities of labor and quantities of other factors of production. Models with one worker per firm or with a linear relationship between labor quantity and output cannot speak to the determinants of a firm's capital intensity or its manager's span of control.

The *matching* of workers to technologies within an industry is the focus of work by Yeaple (2005) and Sampson (2012). These authors also assume a production function with constant returns to labor and thus omit interactions between labor and any other factors of production.[2] Similarly, Grossman and Maggi (2000) study the pairing of workers who perform different production tasks, but in a context with exactly two workers per firm and therefore no scope for variation in factor intensity or firm size. The work of Antràs et al. (2006) does allow for endogenous span of control in a model with matching of workers and managers, but theirs is a one-sector model with international production teams and they assume a particular technology that tightly links the quality and the quantity of labor that a given manager can oversee.

Our analysis extends a familiar trade model with two sectors, two factors, and perfectly-competitive product markets. While most of our analysis assumes frictionless factor markets, we also consider an economy with search and matching frictions. We call one factor "labor" and assume throughout that workers are differentiated along a single dimension that we term "ability." Workers with greater ability are assumed to be more productive in both industries, but the contribution of ability to output may differ across uses. We refer to the second input as "managers," although we might alternatively think of them as "machines." Similar to workers, managers generally differ in ability (or machines differ in quality) and more able managers contribute more to output in both sectors, albeit to an extent that may vary by industry. The modeling of an industry's technology resembles that in Lucas (1978) and the extension provided by Eeckhout and Kircher (2012) to allow for heterogeneity of both factors and decreasing returns to the quantity of one given the quantity of the other. With this formulation, we can address how the economy matches a fixed but heterogenous supply of one input (managers or machines) with a fixed but heterogeneous supply of another (labor) in a setting where the relative number of workers per manager is a matter for firms to decide.

In the next section, we lay out our basic model of an open economy with two countries, two competitive industries, and two heterogeneous factors of production. Section 3 considers trade between countries that have heterogeneous workers but homogeneous managers. Our analysis of this simpler setting aids in understanding the more general case discussed in Sections 4, where managers also are assumed to vary in ability. We show that, with homogeneous managers, the sorting of workers is guided by a cross-industry comparison of the ratio of the elasticity of output with respect to labor quality to the elasticity of output with respect to labor quantity. This can

[2] Both of these authors assume, however, that firms produce differentiated products in a world of monopolistic competition, so that inputs of additional labor by a firm do generate decreasing returns in terms of *revenue*. Thus, these models do share some features with the ones that we study below.

generate a simple sorting pattern in which all the best workers with ability above some threshold level are employed in one sector and the remaining workers are employed in the other. But it also can generate more complex patterns in which, for example, the most able and least able workers sort to one sector while workers with intermediate levels of ability are allocated to the other. Trade between countries with similar distributions of worker talent is determined by their aggregate factor endowments as in the Heckscher-Ohlin model, whereas trade between countries with similar relative endowments reveals a comparative advantage for a country with a superior distribution of labor quality (as reflected in a proportional rightward shift of its talent distribution) in the good produced by the industry in which worker ability contributes more elastically to productivity. With homogeneous managers, relative price movements do not affect within-sector relative wages and therefore have no effect on wage inequality *within industries*. Across industries, the impact of trade on wages reflects a blend of Stolper-Samuelson and Ricardo-Viner forces, as in models with imperfect factor mobility such as Mussa (1982) and Grossman (1983).

In Section 4 we turn to the more interesting case that arises when both factors are heterogeneous and there are complementarities between the qualities of the two factors. If the productivity of a production unit is a strictly log supermodular function of the ability levels of the manager and the workers, there is positive assortative matching of factors in each sector. That is, among the sets of workers and managers that sort to a given sector, the better workers are matched with the better managers. We provide sufficient conditions under which all of the workers with ability above some threshold level and all the managers with ability above some (different) threshold level sort to the same sector. We also provide conditions under which the high-ability workers sort to the same sector as the low-ability managers. More complex sorting patterns are possible as well. When countries share the same distributions of abilities and the sorting patterns do involve a single threshold for each factor, then the country endowed with more managers per worker must export the manager-intensive good.

When there are strong complementarities between the types of workers and managers, the effects of trade or trade liberalization on the wage distribution are subtle and interesting. An increase in the relative price of some good might worsen the matches for all workers and improve the matches for all managers, or vice versa. Alternatively, a change in relative price might improve the matches for workers in one industry while worsening those for workers in the other. We identify conditions for these various shifts in the matching functions and discuss their implications for factor rewards. In particular, we identify situations in which trade causes within-industry income inequality to rise or fall and we show that the impact of trade on an input's within-sector earnings inequality can differ from the changes that occur across sectors.

In Section 5, we extend the analysis to include economies with labor-market frictions by assuming that workers engage in directed search. In this setting, each potential worker seeks a job at a firm of his choosing and manages to be hired by that firm with a probability that depends on the number of applicants per vacancy. We show that, with these search frictions, wage and employment rates both vary with ability; more able workers not only earn higher wages but also enjoy better job prospects. Moreover, we find that trade affects the inequality in expected wages and in employment rates similarly.

Section 6 contains some concluding remarks.

2 The Economic Environment

We examine a world economy comprising two countries, two industries, and two factors of production. We call one of the factors "labor" and refer to individuals as "workers." Each country is endowed with a continuum of workers of various types. The exogenous supply of workers of type q_L in country c is $\bar{L}^c\phi_L^c(q_L)$ for $c = \{A, B\}$, where $\bar{L}^c$ is the aggregate endowment of labor and $\phi_L^c(q_L)$ is the probability density function (pdf) over worker types. For ease of exposition, we assume throughout that $\phi_L^c(q_L)$ is continuous and strictly positive on a finite support $S_L^c = [q_{L\min}^c, q_{L\max}^c]$. We refer to the second factor as "managers," although we could as well have used the term "machines." Country c has a continuum of managers of measure $\bar{H}^c$. We begin in Section 3 by assuming that all managers (or machines) are alike, so that the set of manager types S_H^c has a single element. Subsequently, we introduce manager heterogeneity and then denote the supply of managers of type q_H by $\bar{H}^c\phi_H^c(q_H)$, with $\phi_H^c(q_H)$ continuous and strictly positive on a finite support $S_H^c = [q_{H\min}^c, q_{H\max}^c]$.[3]

Firms in the two countries have access to identical, constant-returns-to-scale technologies. The output generated in an industry by a production unit comprising one manager and a group of workers reflects the *number* of workers that is combined with the manager and the *types of the employed factors.* We could begin by specifying output as a function of the type of the manager and a list of the types of all workers used in the production unit. However, in many models of a manager's "span of control," such as Sattinger (1975), Garicano (2000), and Eeckhout and Kircher (2012), firms have no incentive to combine a given manager with a group of workers of different types.[4] We build on the latter and, to conserve on notation, simply assume that each firm combines a manager of some given type q_H with a group of ℓ workers of a common type q_L, thereby generating output of

$$x_i = \psi_i(q_H, q_L)\,\ell^{\gamma_i}, \;\; 0 < \gamma_i < 1. \tag{1}$$

Here, γ_i is a parameter that reflects the diminishing returns to combining more workers with a given manager and $\psi_i(q_H, q_L)$ is a strictly increasing, twice continuously differentiable, log supermodular function that captures the complementarities between the types of the two factors.[5] We assume that

[3] We focus on an environment where factor endowments are invariant to trade. This makes our results comparable to most previous studies. Future work might consider adjustments in factor endowments - e.g., taking the terminology of workers and managers literally one might study long-run skill acquisition that turns workers into managers. Or, thinking of the second factor as machinery, one might incorporate investment in capital of different qualities.

[4] For example, the manager may be endowed with a unit of time that she must allocate among the various workers, such that each worker's productivity increases with the time devoted to him by the manager, albeit with diminishing returns. The key assumption here is that there is no teamwork or synergy between workers in a firm; they interact only insofar as they compete for the manager's time. See Eeckhout and Kircher (2012) for further discussion. They show, in such a setting, that it is optimal for every firm to combine a manager of some type with a group of workers that share a common type.

[5] We adopt a Cobb-Douglas specification for factor quantities in order to simplify the analysis. Some of our results

factor type contributes to productivity in qualitatively the same way in both sectors and, without further loss of generality, order the types so that $\partial\psi_i/\partial q_F > 0$ for $i = 1, 2$ and $F = H, L$. With this labeling convention, we can refer to q_F as the "ability" of factor F. Note that the industries generally differ in the strength of the complementarities between factors, in the contributions of factor abilities to productivity, and in their factor intensities.

The rest of the model is familiar from neoclassical trade theory. Consumers worldwide share identical and homothetic preferences. Firms hire workers and managers on frictionless national factor markets and engage in perfect competition on integrated world product markets. Countries trade freely, with balanced trade. Note that we neglect for now the search frictions that are a realistic and interesting feature of many markets with heterogeneous factors. We shall extend the analysis to incorporate such frictions in Section 5 below.

3 Homogeneous Managers

We are ultimately interested in the sorting and matching of two heterogeneous factors of production. However, before we get to that, we consider a simpler environment in which one of the factors (managers) has a uniform type. By examining a setting with managers of similar ability, we can gain insight into the sorting of the heterogeneous workers to industries without needing to concern ourselves with the matching of managers and workers. We will introduce manager heterogeneity in Section 4 below.

Suppose that all managers are identical and assume without further loss of generality that their common ability level is $q_H = 1$. Let $\tilde{\psi}_i(q_L) \equiv \psi_i(q_L, 1)$ be the productivity in sector i of workers of ability q_L when combined with any manager who might be employed there. Output per manager in sector i can now be written as $x_i = \tilde{\psi}_i(q_L)\ell^{\gamma_i}$, considering the diminishing returns to the number of workers.

A key variable in the analysis will be the ratio of two elasticities that describe a sector's production technology. One elasticity is $\varepsilon_{\tilde{\psi}_i}(q_L) \equiv q_L\tilde{\psi}_i'(q_L)/\tilde{\psi}(q_L)$, which reflects the responsiveness of output to worker *ability* in sector i, holding constant the number of workers per manager. The other elasticity is γ_i, which is the responsiveness of output to labor *quantity*, holding constant the ability of the workers. Let

$$s_L(q_L) \equiv \frac{\varepsilon_{\tilde{\psi}_1}(q_L)}{\gamma_1} - \frac{\varepsilon_{\tilde{\psi}_2}(q_L)}{\gamma_2}$$

be the difference across sectors in these ratios. We assume for now that $s_L(q_L)$ has a uniform sign for all q_L in the domain of the ability distribution and label the industries so that $s_L(q_L) > 0$. More formally, we adopt for the time being the following assumption:

Assumption 1 $S_H = \{1\}$ and $s_L(q_L) > 0$ for all $q_L \in S_L^A \cup S_L^B$.

would remain the same with an arbitrary constant-returns-to-scale production technology provided that there are no factor intensity reversals.

A firm in sector i chooses the number and type of its workers (per manager) to maximize $\pi_i(q_L, \ell) = p_i \tilde{\psi}_i(q_L) \ell^{\gamma_i} - w(q_L)\ell - r$, where p_i is the price of good i, $w(q_L)$ is the wage of a worker with ability q_L, and r is the salary of the representative manager.[6] The firm can solve its profit maximization problem in two stages. First, it calculates the optimal demand (per manager) for workers of ability q_L when the wage of such workers is $w(q_L)$, which yields

$$\ell_i(q_L) = \left[\frac{\gamma_i p_i \tilde{\psi}_i(q_L)}{w(q_L)}\right]^{\frac{1}{1-\gamma_i}}. \tag{2}$$

Substituting this labor demand into the profit function gives an expression for profits that depends only on the ability of the workers in the production unit, namely

$$\tilde{\pi}_i(q_L) = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \tilde{\psi}_i(q_L)^{\frac{1}{1-\gamma_i}} w(q_L)^{-\frac{\gamma_i}{1-\gamma_i}} - r, \tag{3}$$

where $\bar{\gamma}_i \equiv \gamma_i^{\frac{\gamma_i}{1-\gamma_i}}(1-\gamma_i)$. In the second stage, the firm chooses q_L to maximize $\tilde{\pi}_i(q_L)$. To characterize this optimal choice, let Q_{Li} be the set of abilities of workers that sort into sector i and let Q_{Li}^{int} be the interior of this set. Since the equilibrium wage function must be everywhere continuous, strictly increasing, and differentiable at all points in Q_{Li}^{int}, $i = 1, 2$, the first-order condition of the second-stage maximization problem implies

$$\frac{\varepsilon_{\tilde{\psi}_i}(q_L)}{\gamma_i} = \varepsilon_w(q_L) \text{ for all } q_L \in Q_{Li}^{int}, \tag{4}$$

where $\varepsilon_w(q_L)$ is the elasticity of the wage schedule with respect to ability.[7]

Evidently, firms in sector i choose their workers so that the elasticity of output with respect to ability divided by the elasticity of output with respect to quantity is just equal to the elasticity of the wage schedule.[8] If (4) were to hold at just one value of $q_L \in Q_{Li}$, then all firms in industry i would demand workers with the same ability level. Of course, such an outcome would not be consistent with full employment of all types of workers. Instead, (4) must hold for all $q_L \in Q_{Li}^{int}$. In such circumstances, the firms in sector i are indifferent among the various types of workers that are employed in the sector. This indifference incorporates not only the heterogeneous productivities of the different workers, but also the optimal adjustment in the number of workers that the firm would make were it to switch from one type of worker to another. The accompanying adjustment

[6] We suppress for now the country superscript c, because we focus on firms' decisions in a single country.

[7] The strict monotonicity of the wage function follows from the strict monotonicity of the productivity functions $\tilde{\psi}_i(q_L)$; if wages were declining over some range of abilities, all firms would prefer to hire the most able workers in this range. The continuity of the wage function follows from the continuity of the productivity function; if wages were to jump at some q_L', firms would strictly prefer workers with ability a shade below q_L' to workers with ability a shade above q_L', because the former would be only slightly less productive but would cost discretely less. In the appendix, we prove that the wage function must be differentiable in the interior of Q_{Li} for the more general case in which managers are heterogeneous; that proof applies as well to the case with homogeneous managers that we consider in this section.

[8] Note that Costinot and Vogel (2010) derive a similar wage schedule, except that $\gamma_i = 1$ for all i for their economy with linear output.

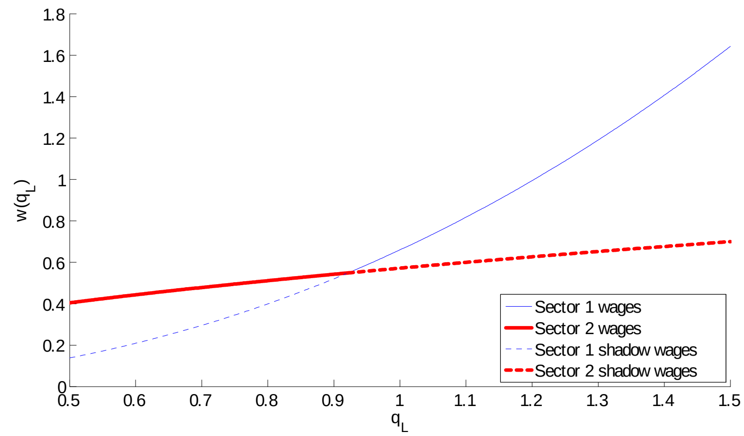

Figure 1: Wages of workers: homogeneous managers

in quantity explains why it is the ratio of the two elasticities—and not just the responsiveness of output to ability—that firms take into account when they contemplate a change in the ability levels of their employees.

The requirement that the wage function has an elasticity $\varepsilon_{\tilde{\psi}_i}(q_L)/\gamma_i$ for all worker types that are hired in sector i is equivalent to the requirement that the wage function takes the form

$$w\left(q_L\right) = w_i \tilde{\psi}_i \left(q_L\right)^{1/\gamma_i} \text{ for } q_L \in Q_{Li} \ , \tag{5}$$

for some (endogenous) wage anchor, w_i. This wage function dictates the sorting pattern for labor. Consider any worker type, say q_L^*, that is hired in equilibrium by both sectors and is thus paid the same wage in both. Under Assumption 1, workers with ability greater than q_L^* can earn more in sector 1 than in sector 2, because the wage that makes firms indifferent between these more able workers and workers of ability q_L^* is higher there. Similarly, workers with ability less than q_L^* face better prospects in sector 2, because firms there are more willing to sacrifice ability after taking account of the optimal adjustment in quantity. It follows that the equilibrium sorting pattern has a single cutoff level q_L^* such that workers with ability above q_L^* are employed in sector 1 and those with ability below q_L^* are employed in sector 2.

Figure 1 shows the qualitative features of the equilibrium wage schedule. The solid curve depicts what workers of different abilities actually are paid in equilibrium, considering that those with ability $q_L \geq q_L^*$ are employed in sector 1 and those with ability $q_L \leq q_L^*$ are employed in sector 2. The broken curves show what the wages for different types of workers would have to be in order to make firms in an industry indifferent between hiring these types and hiring the types of workers that actually are employed in the industry. From now on, we will refer to these wages that reflect what firms in the opposite sector would be willing to pay to replace their actual employees with these alternative hires as the "shadow wages." Notice that the shadow wages are less than the equilibrium wages paid to workers in their actual sector of employment, as of course they must be.

Notice too that firms in either sector are willing to hire the workers with the marginal ability q_L^*.

We record our observations about the equilibrium sorting pattern in

Proposition 1 *Suppose that Assumption 1 holds. Then, in any competitive equilibrium with employment in both sectors, the more able workers with $q_L \geq q_L^*$ are employed in sector 1 and the less able workers with $q_L \leq q_L^*$ are employed in sector 2, for some $q_L^* \in S_L$.*

The intuition for this sorting pattern should be apparent by now. Sorting is determined by comparing across sectors the ratios $\varepsilon_{\tilde{\psi}_i}/\gamma_i$. On the one hand, when $\varepsilon_{\tilde{\psi}_i}$ is large, there is a big return to moving higher *ability* workers to sector i inasmuch as marginal ability contributes greatly to productivity there. On the other hand, when γ_i is large, output in sector i expands rapidly with the *number* of employed workers, irrespective of their ability. In such circumstances, it makes economic sense to deploy relatively large numbers of workers in the industry, and this can be accomplished at lower cost by hiring the workers of lesser ability. The equilibrium sorting pattern reflects a trade-off between the returns to ability and the returns to quantity.

We can now record the remaining equilibrium conditions by invoking labor-market clearing for the various types of workers, imposing continuity of the wage function at q_L^*, and adding a requirement that all active firms must break even. We focus henceforth on equilibria characterized by incomplete specialization, which arise whenever the endowment ratios and skill distribution are not too extreme. Consider first the aggregate supply and demand for workers with ability greater than q_L^*. Define $e_i(q_L) = \tilde{\psi}_i(q_L)^{1/\gamma_i}\,\ell(q_L)$ as the "effective labor" hired per manager by a firm that employs workers with ability q_L. Such a firm produces $[e_i(q_L)]^{\gamma_i}$ units of good i for every manager it employs. Using the expression for labor demand (2) and considering the wage schedule (5), every firm operating in sector i combines the same amount of effective labor with any one of its managers, namely $e_i = (\gamma_i p_i/w_i)^{1/(1-\gamma_i)}$. It follows that the firms operating in sector i collectively demand $H_i e_i = H_i(\gamma_i p_i/w_i)^{1/(1-\gamma_i)}$ units of effective labor, where H_i is the measure of managers employed in sector i. The total supply of effective labor is simply the measure of effective units of labor among those that sort to sector i. Equating demand and supply gives

$$H_i\left(\frac{\gamma_i p_i}{w_i}\right)^{\frac{1}{1-\gamma_i}} = \bar{L}\int_{q_L\in Q_{Li}} \tilde{\psi}_i(q_L)^{1/\gamma_i}\phi_L(q_L)\,dq_L, \text{ for } i=1,2.$$

Proposition 1 tells us which workers are employed in which sectors, i.e., $Q_{L1} = [q_L^*, q_{L\max}]$ and $Q_{L2} = [q_{L\min}, q_L^*]$. So we can write

$$H_1\left(\frac{\gamma_1 p_1}{w_1}\right)^{\frac{1}{1-\gamma_1}} = \bar{L}\int_{q_L^*}^{q_{L\max}} \tilde{\psi}_1(q_L)^{1/\gamma_1}\phi_L(q_L)\,dq_L \tag{6}$$

and

$$\left(\bar{H}-H_1\right)\left(\frac{\gamma_2 p_2}{w_2}\right)^{\frac{1}{1-\gamma_2}} = \bar{L}\int_{q_{L\min}}^{q_L^*} \tilde{\psi}_2(q_L)^{1/\gamma_2}\phi_L(q_L)\,dq_L \tag{7}$$

where, in (7), we have used the market-clearing condition for managers, $H_1 + H_2 = \bar{H}$.

The wage function must be continuous at q_L^*, or else the firms that hire these workers in sector 1 could save discretely on labor costs by downgrading their workforce slightly, while sacrificing only a negligible quantity of output. Continuity of the wage schedule at q_L^* implies that

$$w_1 \tilde{\psi}_1 (q_L^*)^{1/\gamma_1} = w_2 \tilde{\psi}_2 (q_L^*)^{1/\gamma_2}. \tag{8}$$

Finally, profits must be equal to zero for firms operating in both sectors, assuming that the economy is incompletely specialized (otherwise they are zero in the active sector and potentially negative in the other). These requirements together with (3) pin down the equilibrium salary for managers, $r = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \tilde{\psi}_i (q_L)^{\frac{1}{1-\gamma_i}} w(q_L)^{-\frac{\gamma_i}{1-\gamma_i}}$, and also ensure that

$$\bar{\gamma}_1 p_1^{\frac{1}{1-\gamma_1}} w_1^{-\frac{\gamma_1}{1-\gamma_1}} = \bar{\gamma}_2 p_2^{\frac{1}{1-\gamma_2}} w_2^{-\frac{\gamma_2}{1-\gamma_2}}. \tag{9}$$

Equations (6)-(9) jointly determine the marginal worker q_L^*, the wage anchors w_1 and w_2, and the measure of managers H_1 employed in sector 1 for any economy that produces positive amounts of both goods. The equilibrium salary of managers is given by

$$r = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} w_i^{-\frac{\gamma_i}{1-\gamma_i}}, \ i = 1, 2. \tag{10}$$

In what follows, we are interested in the determinants of the trade pattern between countries that differ in their relative endowments of labor to managers and in their distributions of worker ability, and especially in how trade between such countries affects their distributions of income.

3.1 Determinants of the Trade Pattern

Consider two countries that trade freely at common world prices but that differ in some way in their factor supplies. Since consumers have identical and homothetic tastes worldwide, the trade pattern between them can be identified by examining the countries' relative outputs of the two goods at common prices. Accordingly, we investigate how a change in parameters reflecting factor endowments affects relative outputs of the two goods at given prices.

In each country, a firm in industry i employs $e_i = (\gamma_i p_i / w_i)^{1/(1-\gamma_i)}$ units of effective labor per manager, thereby producing $e_i^{\gamma_i}$ units of good i. Thus, aggregate output in sector i is

$$X_i = H_i \left(\frac{\gamma_i p_i}{w_i} \right)^{\frac{\gamma_i}{1-\gamma_i}}, \quad i = 1, 2. \tag{11}$$

We can substitute the equal-profit condition (9) into this expression to eliminate the wage anchors.

Taking the ratio of the resulting expressions yields[9]

$$\frac{X_1}{X_2} = \frac{H_1}{\left(\bar{H} - H_1\right)} \frac{(1-\gamma_2)p_2}{(1-\gamma_1)p_1},$$

which implies that the relative output of good 1 is greater in whichever country allocates the greater share of its managers to producing that good.

3.1.1 Relative Factor Endowments

First, suppose the two countries have the same distributions of worker ability but differ in their relative aggregate endowments, $\bar{H}/\bar{L}$. To find the pattern of trade, we totally differentiate the four-equation system comprising (6)-(9) with respect to $\bar{H}/\bar{L}$ and examine how a change in relative endowments affects the allocation of managers to sector 1. The algebra in the appendix establishes the following proposition.

Proposition 2 *Suppose that Assumption 1 holds and that* $\phi_L^A(q_L) = \phi_L^B(q_L)$ *for* $q_L \in S_L^A = S_L^B$*. Then country* A *exports the manager-intensive good if and only if* $\bar{H}^A/\bar{L}^A > \bar{H}^B/\bar{L}^B$*.*

Proposition 2 represents, of course, an extension of the Heckscher-Ohlin theorem. When worker talent is distributed similarly in the two countries, the sorting of workers to sectors generates no comparative advantages and so has no independent bearing on the trade pattern. Comparative advantage is governed instead by relative quantities of the factors, just as in the case of homogeneous labor. Of course, the heterogeneous workers do differ in their suitability for employment in the two sectors, which means that supply elasticities will reflect the difference across sectors in the elasticity ratio $\varepsilon_{\tilde{\psi}_i}(q_L)/\gamma_i$ and so too will the volume of trade.

3.1.2 Distributions of Labor Ability

Now suppose that the relative number of managers and workers is the same in the two countries, but that country A has relatively better workers in the sense that the pdf for worker ability in country A is a rightward shift (RS) of the similar function in country B. That is,

$$\phi_L^B(q_L/\lambda) = \phi_L^A(q_L) \quad \text{for all } q_L \in S_L^A, \text{ for some } \lambda > 1, \tag{12}$$

which has the interpretation that every worker in country A is λ times as productive as his counterpart in the talent distribution in country B. Again, we need to totally differentiate the system of equations (6)-(9) in order to identify the impact of a rightward shift in the talent distribution on employment of managers in sector 1. The algebra in the appendix supports the following conclusion.

[9] This condition can alternatively be derived from the observation that in sector i the fraction $1-\gamma_i$ of revenue is paid to managers, i.e., $(1-\gamma_i)\, p_i X_i = rH_i$.

Proposition 3 *Suppose that Assumption 1 holds, that* $\bar{H}^A/\bar{L}^A = \bar{H}^B/\bar{L}^B$, *and that* $\phi_L^A(q_L)$ *is a rightward shift of* $\phi_L^B(q_L)$ *for some* $\lambda > 1$. *If* $\varepsilon_{\tilde{\psi}_i}(q'_L) > \varepsilon_{\tilde{\psi}_j}(q''_L)$ *for all* $q'_L, q''_L \in S_L^A \cup S_L^B$, $i \neq j$, $i, j \in \{1,2\}$, *then country* A *exports good* i.

The proposition states that the country that has the superior labor force exports the good produced in the industry where worker ability contributes more elastically to productivity. Notice that this need not be the good produced by the country's most able workers inasmuch as sorting reflects the ranking of $\varepsilon_{\tilde{\psi}_1}(q_L)/\gamma_1$ versus $\varepsilon_{\tilde{\psi}_2}(q_L)/\gamma_2$, whereas the trade pattern depends only on the ranking of $\varepsilon_{\tilde{\psi}_1}(q_L)$ versus $\varepsilon_{\tilde{\psi}_2}(q_L)$. This result can be understood by thinking about the sources of comparative advantage in this setting. With $\bar{H}^A/\bar{L}^A = \bar{H}^B/\bar{L}^B$, the cross-sectoral difference in factor intensity is not a source of comparative advantage for either country. Meanwhile, with $\varepsilon_{\tilde{\psi}_1}(q_L)$ different from $\varepsilon_{\tilde{\psi}_2}(q_L)$, worker ability contributes differently to productivity in the two sectors. Country A, which is relatively better endowed with more able workers, enjoys a technological comparative advantage in the industry in which ability matters more for output.[10]

3.2 The Effects of Trade on Income Distribution

We are especially interested in the relationship between trade and income distribution in a world with heterogeneous factors of production. As in other neoclassical trade models, commodity prices mediate the link between trade and earnings. The opening of trade (or subsequent trade liberalization) generates an increase in the relative price of a country's export good, which in turn alters the demand for different factors and factor types. Accordingly, we examine the comparative static response of the wage schedule and managerial salaries to a change in the relative price of the final goods.

Note first that the wage function (5) pins down the relative wages of the various workers who are employed in a given sector. A change in relative price can alter the relative pay only of workers who are initially or ultimately employed in different industries . The calculations in the appendix establish the following findings.[11]

Proposition 4 *Suppose that Assumption 1 holds. Then when* $\hat{p}_1 > 0$, *(i)* $\hat{w}_1 > \hat{w}_2$; *(ii) if* $\gamma_1 \approx \gamma_2$, *then* $\hat{w}_1 > \hat{p}_1 > \hat{r} > 0 > \hat{w}_2$; *(iii) if* $\gamma_1 > \gamma_2$ *and* $s_L(q_L^*) \approx 0$, *then* $\hat{w}_1 \approx \hat{w}_2 > \hat{p}_1 > 0 > \hat{r}$; *and (iv) if* $\gamma_1 < \gamma_2$ *and* $s_L(q_L^*) \approx 0$, *then* $\hat{r} > \hat{p}_1 > 0 > \hat{w}_1 \approx \hat{w}_2$.

Part (i) of Proposition 4 says that any worker employed in industry 1 gains relative to any worker employed in industry 2 when the relative price of good 1 rises. The remaining parts of the proposition capture the two distinct influences on factor returns in an economy with heterogeneous

[10] In the special case in which $\tilde{\psi}_i(q_L)$ is a power function for $i = 1,2$, i.e., $\tilde{\psi}_i(q_L) = a_i q_L^{\alpha_i}$ for some $a_i, \alpha_i > 0$, $\varepsilon_{\tilde{\psi}_i}(q'_L) > \varepsilon_{\tilde{\psi}_j}(q''_L)$ for all q'_L and q''_L if and only if $\alpha_i > \alpha_j$. Moreover, in this case, $s_L(q_L) > 0$ for all q_L if and only if $\alpha_1/\gamma_1 > \alpha_2/\gamma_2$. Evidently, the conditions of Proposition 3 are easily satisfied. When $\tilde{\psi}_i(q_L)$ is not a power function for $i = 1,2$, the requirement that $\varepsilon_{\tilde{\psi}_i}(q'_L) > \varepsilon_{\tilde{\psi}_j}(q''_L)$ for all $q'_L, q''_L \in S_L^A \cup S_L^B$, $i \neq j$, $i, j \in \{1,2\}$ is not trivial, but it can be weakened into a comparison of the average elasticities of productivity with respect to ability in the two sectors. See the proof of Proposition 3 in the appendix.

[11] In what follows, we use a "hat" over a variable to indicate an incremental, proportional change; i.e., $\hat{z} = dz/z$.

labor. The cross-sectoral difference in factor intensities introduces a force akin to that in the standard Heckscher-Ohlin model with homogeneous labor, whereby real wages tend to rise and real managerial salaries tend to fall if the sector experiencing the increase in relative price is the more labor intensive of the two. But the heterogeneity of labor implies that different workers are not equally proficient as potential employees in the two sectors, which introduces a force akin to that in the Ricardo-Viner model (see, e.g., Jones, 1971). Indeed, our result is reminiscent of findings in a model with "imperfect factor mobility" (Mussa, 1982) or "partially mobile capital" (Grossman, 1983). That is, if the factor intensity difference across industries is large (i.e., $\gamma_1 \neq \gamma_2$) and the force for inter-industry sorting of the marginal worker types is muted (i.e., $s_L(q_L^*) \approx 0$), then all types of the factor used intensively in sector 1 must gain, while all types of the factor used intensively in sector 2 must lose (parts (iii) and (iv) of the proposition). On the other hand, if the factor intensity difference is small (i.e., $\gamma_1 \approx \gamma_2$) and the different marginal worker types are imperfect substitutes in the two sectors ($s_L(q_L^*) > 0$), then all workers initially employed in the expanding sector will gain, all workers that continue to be employed in the contracting sector will lose, and the effect on the well being of managers will depend on their consumption pattern (part (ii) of the proposition). In the former case, managerial salaries rise in terms of the import good but fall in terms of the export good, as in the Ricardo-Viner model with mobile managers and sector-specific labor. In the latter case, real managerial salaries rise if the export sector is manager intensive and fall if it is labor intensive, as in the Heckscher-Ohlin model.

In summary, when managers are homogeneous and thus matching is indeterminate, trade has no effect on within-industry wage inequality; the relative earnings of any pair of workers employed in the same sector is pinned down by (5).[12] Meanwhile, an increase in the price of good 1 raises the wage anchor in sector 1 relative to the wage anchor in sector 2 (see part (i) of the proposition). And since the higher-ability, higher-wage workers are employed in sector 1, this implies that by raising wages in sector 1 relative to wages in sector 2 an increase in the price of good 1 increases overall wage inequality in country A, while reducing wage inequality in country B. These results provide a benchmark for comparing those in Section 4, where market forces determine the matching of workers with heterogeneous managers and where trade can affect income distribution by altering the pattern of matches.

3.3 Sorting Reversals

So far, we have used Assumption 1 to characterize the sorting of heterogeneous workers and the resulting trade structure. In this final part of the section on homogeneous managers, we clarify what can happen when $s_L(q_L)$ switches sign.

First note that if $\tilde{\psi}_i(q_L)$ is a power function for $i = 1, 2$, the function $s_L(q_L)$ does not depend on q_L inasmuch as the elasticities of productivity with respect to ability then are constants. In such

[12] We should perhaps mention that, by altering the composition of workers in each sector, trade will affect any measure of within-industry wage inequality (such as, for example, the Gini coefficient) that does not hold the set of workers in the comparison fixed.

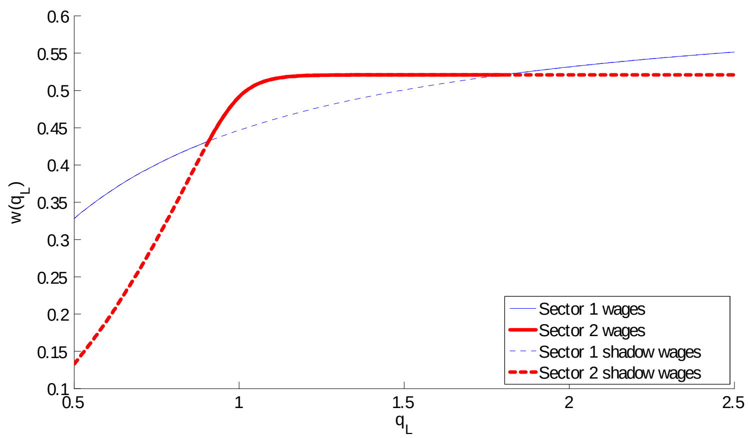

Figure 2: Wages with a reversal of sorting

circumstances, $s_L(q_L)$ is either always positive or always negative, and we can assume $s_L(q_L) > 0$ without loss of generality, because this only amounts to a particular labeling of the sectors. However, when $\tilde{\psi}_i(q_L)$ is not a power function for some i, the assumption that $s_L(q_L)$ has a uniform sign for all $q_L \in S_L$ imposes meaningful restrictions on the forms of the productivity functions and the support of the distribution of worker talent. Without these restrictions, we cannot be sure that the most able workers sort into one sector and the least able workers sort into the other.

To illustrate what can happen when $s_L(q_L)$ changes signs, suppose that the productivity of a firm in sector i that hires workers of ability q_L is given by

$$\tilde{\psi}_i(q_L) = \left(\alpha_i q_L^{\rho_i} + 1\right)^{1/\rho_i}, \ \alpha_i > 0, \ \rho_i < 0 \ \text{ for } i = 1, 2. \tag{13}$$

This specification implies a constant elasticity of substitution between the ability of workers and the ability of managers in generating the productivity of the firm, and that worker and managerial ability are, in fact, complements. Of course, with homogeneous managers, firms have no possibility to adjust manager type in order to take advantage of this complementarity. Nonetheless, the CES specification for productivity represents a legitimate and even a plausible functional form.

When productivity takes the form indicated in (13), the elasticity of productivity with respect to worker ability in sector i is given by $\varepsilon_{\tilde{\psi}_i}(q_L) = \alpha_i q_L^{\rho_i} / \left(\alpha_i q_L^{\rho_i} + 1\right)$. If $\rho_1 \neq \rho_2$ then $\varepsilon_{\tilde{\psi}_1}(q_L) - \varepsilon_{\tilde{\psi}_2}(q_L)$ necessarily switches signs on $q_L \in [0, +\infty)$ and therefore $s_L(q_L)$ may switch signs on the support of the distribution of worker ability, depending on the industry factor intensities and the range of the talent distribution.

Figure 2 depicts an equilibrium wage schedule for an economy in which $s_L(q_L) < 0$ for low values of q_L and $s_L(q_L) > 0$ for high values of q_L.[13] In this economy, the most and least able workers sort to sector 1 while a middle range of workers is hired into sector 2. The thin solid curves in the figure depict the wages of workers employed in sector 1 as a function of their ability, while

[13] See Lim (2013) for the functional forms and parameter values that were used to generate this figure.

the thick solid curve depicts the wages of workers employed in sector 2. The broken thin curve depicts the shadow wage for workers in sector 2, i.e., the wage offers they could garner were they to seek jobs in sector 1. Similarly, the broken thick curve depicts the shadow wages available in sector 2 for workers actually employed in sector 1. Clearly, each worker sorts into the industry that offers him the highest wage.

Figure 2 represents an economy in which $\gamma_1 = \gamma_2 = 0.5$, i.e., the industries have similar factor intensities. However, $\rho_1 \neq \rho_2$, which generates the different elasticities of productivity at different levels of ability. The comparative statics reveal an interesting response of wages to relative price changes for these parameter values. Inasmuch as the factor intensities are common to the two industries, there are no Stolper-Samuelson forces at work. But the workers that sort to sector 1 are better suited for employment there than their counterparts working in sector 2. The forces akin to those in a Ricardo-Viner model imply that when p_1 rises, the real wages of all workers employed in sector 1 also rise, while the real wages of all workers employed in sector 2 decline. In short, an increase in the relative price of good 1 generates income gains for workers with high or low wages but income losses for those in the middle of the wage distribution.[14]

When the two sectors differ in their factor intensities, the Stolper-Samuelson forces will again play a role in determining the effects of trade on the wage distribution. Take, for example, a case in which $\gamma_1 = 0.9$ and $\gamma_2 = 0.1$, so that sector 1 is much more labor intensive than sector 2. We have solved this example numerically for various sets of the other parameter values.[15] In all such cases, we found that an increase in the price of good 1 raises both wage anchors more than in proportion to the price change, so that all workers gain in real income. Meanwhile, the salary of managers falls. These results are familiar from the Stolper-Samuelson theorem, and they are similar to what we found with great disparities in factor intensities for economies that satisfy Assumption 1. We find as well that an increase in p_1 benefits workers employed in sector 1 more than those employed in sector 2, in keeping with our observations that workers are partially specific to their industry of employment due to comparative productivity differences.[16] Price changes do not affect relative wages for workers employed in the same industry, even if those workers are at opposite tails of the talent distribution as is the case for some pairs of workers that sort to sector 1.

[14] For this example, we calculate that a 5% increase in p_1 raises the wage anchor w_1 by 5.7%, while depressing the wage anchor w_2 by 4.2%. Managers' salaries rise by 4.3%, which is proportionately less than the increase in price.

[15] As one example, we have solved the model for the case in which world prices are $(p_1, p_2) = (1, 1)$ and the economy has an aggregate endowment of $(\bar{H}, \bar{L}) = (1, 1)$. In this example, we assumed that worker ability is drawn from a truncated Pareto distribution on the support $S_L = [0.8, 1.8]$ with the shape parameter 3, and that the technological parameters are given by $(\gamma_1, \alpha_1, \rho_1) = (0.9, 0.7, -1)$ and $(\gamma_2, \alpha_2, \rho_2) = (0.1, 0.3, -20)$. In the computed equilibrium, sector 2 employs workers with $q_L \in [1.034, 1.211]$ and 0.953 managers. The wage anchors are $w_1 = 0.718$ and $w_2 = 0.434$ while the managers earn a salary of $r = 0.765$.

[16] Using the parameter values detailed in the previous footnote, we find that a 5% increase in the price p_1 generates a wage hike of 5.6% for workers employed in sector 1, a wage hike of 5.4% for workers employed in sector 2, and a slary reduction of 0.6% for all managers.

4 Heterogeneous Managers

We now introduce manager heterogeneity and model the diversity of manager types similarly to that for workers. More specifically, we posit a mass $\bar{H}^c$ of managers in country c and a probability density $\phi_H^c(q_H)$ of managers with ability q_H for $q_H \in S_H^c = [q_{H\min}^c, q_{H\max}^c]$. We take the supply of managers and their ability distribution as given throughout the analysis.

To capture complementarities between workers and managers within a production unit, we take the productivity functions $\psi_i(q_H, q_L)$ for $i = 1, 2$ to be log supermodular; i.e., worker ability contributes relatively more to productivity when the better workers are teamed with a more able manager than when they are teamed with a less able manager. For convenience, we also assume that $\psi_i(q_H, q_L)$ is strictly increasing and twice continuously differentiable. In such circumstances, log supermodularity of the productivity functions implies that $\psi_{iH}(q_H, q_L)/\psi_i(q_H, q_L)$ is increasing in q_L and $\psi_{iL}(q_H, q_L)/\psi_i(q_H, q_L)$ is increasing in q_H, where $\psi_{iF}(q_H, q_L)$ is the partial derivative of $\psi_i(q_H, q_L)$ with respect to q_F, $F = H, L$. For the most part, we shall focus on the case where $\psi_i(q_H, q_L)$ is *strictly* log supermodular and then we will invoke

Assumption 2 (i) $S_H = [q_{H\min}, q_{H\max}]$, $0 < q_{H\min} < q_{H\max} < +\infty$; (ii) $\psi_i(q_H, q_L)$ is strictly increasing, twice continuously differentiable, and *strictly* log supermodular for $i = 1, 2$.

However, we will also make occasional reference to the case of *weak* log supermodularity that arises when the productivity functions are multiplicatively separable in their two arguments.[17]

We can proceed as before by treating the firm's profit maximization problem in stages. First, the firm takes as given the ability of its workers and of its manager and chooses the size of its production team. This yields the labor demand per manager as a function of the employee types, namely

$$\ell(q_L, q_H) = \left[\frac{\gamma_i p_i \psi_i(q_H, q_L)}{w(q_L)}\right]^{\frac{1}{1-\gamma_i}}.$$

Substituting this expression for labor demand into that for firm profits yields

$$\tilde{\pi}_i(q_H, q_L) = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \psi_i(q_H, q_L)^{\frac{1}{1-\gamma_i}} w(q_L)^{-\frac{\gamma_i}{1-\gamma_i}} - r(q_H). \tag{14}$$

Next, the firm identifies the most suitable workers to combine with the manager, taking the continuous and strictly increasing wage schedule as given.[18] This yields a profit function,

$$\Pi_i(q_H) = \max_{q_L \in S_L} \tilde{\pi}_i(q_H, q_L), \tag{15}$$

for $q_H \in S_H$, $i = 1, 2$. Finally, the firm selects q_H to maximize $\Pi_i(q_H)$, given the continuous and

[17] In particular, we shall refer to a case of "Cobb-Douglas productivity," which arises when the productivity functions are multiplicatively separable and have constant elasticities of productivity with respect to the ability of either factor. In such circumstances, we can write $\psi_i(q_H, q_L) = q_H^{\beta_i} q_L^{\alpha_i}$ for some $\alpha_i > 0$ and $\beta_i > 0$.

[18] The wage schedule must be continuous and strictly increasing for reasons analogous to those that apply with homogeneous managers.

strictly increasing salary schedule, $r(q_H)$. In an equilibrium, firms must be indifferent between the various managers that are employed in an industry, or else all would hire a particular type (or types) and some managers would be unemployed.

We show in the appendix that the solution to the firm's profit-maximization problem and the requirement that firms must be indifferent among the managers that sort to an industry together generate equilibrium allocation sets Q_{Li} and Q_{Hi} that are unions of closed intervals (where Q_{Fi} represents the set of types of factor F that sorts to industry i, for $F = H, L$ and $i = 1, 2$). Moreover, under Assumption 2, there must be positive assortative matching (PAM) of managers and workers *within* each sector. In other words, among the managers and workers that sort to a sector, the better workers are teamed with the better managers.[19] However, as we shall see, PAM need not apply economy-wide.

Let $m_i(q_H)$ denote the solution to (15); i.e., $m_i(q_H)$ is the common ability level of the workers who would be teamed with a manager of ability q_H if that manager happens to be employed in sector i. The equilibrium matching function for the economy, $m(q_H)$, consists of $m_1(q_H)$ for $q_H \in Q_{H1}$ and $m_2(q_H)$ for $q_H \in Q_{H2}$. The matching function generates a pair of closed graphs

$$M_i = [\{q_H, q_L\} \mid q_L \in m_i(q_H) \text{ for all } q_H \in Q_{Hi}] \ , i = 1, 2,$$

where M_i represents the production units that form in sector i in equilibrium. These graphs comprise a union of connected sets M_i^n such that $m_i(q_H)$ is continuous and strictly increasing in each set but may jump discontinuously between them.

Consider an equilibrium with incomplete specialization, so that a positive measure of managers sorts to each sector. All firms that are active in sector i earn zero profits, which implies

$$r(q_H) = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \psi_i [q_H, m_i(q_H)]^{\frac{1}{1-\gamma_i}} w[m_i(q_H)]^{-\frac{\gamma_i}{1-\gamma_i}} \quad \text{for all } q_H \in Q_{Hi}, \ i = 1, 2. \tag{16}$$

Continuity of the wage and salary schedules implies that both functions are differentiable almost everywhere. Moreover, profit maximization and (16) imply that, at all interior points in a connected subset M_i^n of M_i, the salary function $r(\cdot)$ and the wage function $w(\cdot)$ are differentiable; see the appendix for proof. It follows that the solution to (15) must satisfy the first-order condition

$$\frac{m(q_H)\,\psi_{iL}[q_H, m(q_H)]}{\gamma_i \psi_i [q_H, m(q_H)]} = \frac{m(q_H)\, w'[m(q_H)]}{w[m(q_H)]} \tag{17}$$

for all $\{q_H, m(q_H)\} \in M_i^{n,int}$, $\ n \in N_i$, $\ i = 1, 2$, where $M_i^{n,int}$ is the interior of the set M_i^n. Also,

[19] With strict log supermodularity of the productivitiy function $\psi_i(q_H, q_L)$, PAM within sector i follows directly from the arguments in Eeckhout and Kircher (2012). If $\psi_i(q_H, q_L)$ is only *weakly* log supermodular, as in the case of Cobb-Douglas productivity, there always exists an equilibrium with PAM in each industry, although the equilibrium is not unique. Indeed, in this case, the matching of workers and managers in a sector is not well determined by the model. Such indeterminacy reflects that the relative productivity of a team of more able workers compared to a team of less able workers is independent of the type of the manager. However, as we show in the appendix, the indeterminacy of matching in the Cobb-Douglas case does not imply indeterminacy of allocation sets, output levels, or factor prices; these outcomes in fact are unique.

(16) and (17) imply that

$$\frac{q_H \psi_{iH}\left[q_H, m\left(q_H\right)\right]}{\left(1-\gamma_i\right)\psi_i\left[q_H, m\left(q_H\right)\right]} = \frac{q_H r'\left(q_H\right)}{r\left(q_H\right)} \tag{18}$$

for all $\{q_H, m\left(q_H\right)\} \in M_i^{n,int}, \quad n \in N_i, \quad i = 1, 2.$

Notice that the left-hand side of (17) represents a ratio of elasticities, namely the elasticity of productivity in sector i with respect to worker ability divided by the elasticity of output with respect to labor quantity. Profit maximization requires that this ratio be equal to the elasticity of the wage schedule. This is quite analogous to (4) for the case of homogeneous managers, except that now the elasticity of productivity with respect to worker ability reflects the matches that take place in equilibrium and therefore the sorting of managers to sectors. Equation (18) is an analogous condition that leaves firms indifferent among the managers hired in sector i; it equates the quotient of the elasticity of productivity with respect to manager ability and the elasticity of output with respect to the number of managers to the elasticity of the salary schedule.[20]

4.1 Sorting with Heterogeneous Managers

How do workers and managers sort to industries? Recall Figures 1 and 2 in Section 3 that show equilibrium wage and shadow wage functions for the case of homogeneous managers. We argued that the slope of the wage function must be greater just to the right of any boundary point between connected sets of workers that sort to different industries than the slope of the shadow wage function showing what the other industry would be willing to pay. An analogous condition applies when managers are heterogeneous. Let $q_L^\dagger$ be the boundary between some sets of workers that sort to different industries, so that workers with ability just above $q_L^\dagger$ sort to one sector and workers with ability just below $q_L^\dagger$ sort to the other. In equilibrium, the wage function $w\left(q_L\right)$ must be at least as steep to the right of $q_L^\dagger$ as it is to the left; otherwise the firms that employ workers with abilities just above $q_L^\dagger$ could earn greater profits by slightly downgrading their workforce and those that hire workers with abilities below $q_L^\dagger$ could earn earn greater profits by slightly upgrading theirs. A similar argument applies for managers, which implies that the salary function $r\left(q_H\right)$ must be (weakly) steeper just to the right of any boundary point $q_H^\dagger$ than just to the left of such a point.

Now that we have laid out the equilibrium conditions that guide matching and sorting, we turn to the requirements for factor-market clearing. To this end, consider some connected set of managers $[q_{Ha}, q_H]$ that sorts to industry i and the set of workers $\left[m\left(q_{Ha}\right), m\left(q_H\right)\right]$ with whom these managers are matched in equilibrium. A profit-maximizing firm in sector i that employs a manager with ability q_H and workers of ability q_L demands $\ell\left(q_H, q_L\right) = \left[\gamma_i r\left(q_H\right)\right] / \left[\left(1-\gamma_i\right) w\left(q_L\right)\right]$ workers per manager. Since the matching function is everywhere increasing, it follows that

$$\bar{H}\int_{q_{Ha}}^{q_H} \frac{\gamma_i r\left(q\right)}{\left(1-\gamma_i\right) w\left[m\left(q\right)\right]}\phi_H\left(q\right) dq = \bar{L}\int_{m\left(q_{Ha}\right)}^{m\left(q_H\right)} \phi_L\left[m\left(q\right)\right] dq \ ,$$

[20] The technologies exhibit constant returns to scale in the quantities of the two factors, so the elasticity of output with respect to the number of managers in sector i is $1-\gamma_i$.

where the left-hand side represents the total measure of workers demanded by firms operating in sector i that hire managers with ability between q_{Ha} and q_H and the right-hand side represents the measure of workers available to be teamed with those managers. Since the left-hand side is differentiable in q_H as long as q_H is not a boundary point between managers that sort to different industries, this equation implies that the matching function $m(q_H)$ also is differentiable at such points. That being the case, we can differentiate the labor-market clearing condition with respect to q_H to derive a differential equation for the matching function, namely

$$\bar{H}\frac{\gamma_i r(q_H)}{(1-\gamma_i)\, w[m(q_H)]}\phi_H(q_H) = \bar{L}\phi_L[m(q_H)]\, m'(q_H) \tag{19}$$

for $\{q_H, m(q_H)\} \in M_i^{n,int}$, $n \in N_i$, $i = 1,2$. This equation says that the labor demanded by a (small) set of managers with ability in $[q_H, q_H + dq_H]$ equals the density of workers in the economy that match with these managers.

Equations (17), (18) and (19) comprise three differential equations that are satisfied in any competitive equilibrium by the wage schedule $w(q_L)$, the salary schedule $r(q_H)$, and the matching function $m(q_H)$. Together with the zero-profit condition and a set of boundary conditions, these equations can be used to characterize an equilibrium allocation.

Let us define a *threshold equilibrium* as any equilibrium that can be characterized by a pair of boundary points q_L^* and q_H^*, such that all workers with ability less than q_L^* sort to some sector while all workers with ability greater than q_L^* sort to the other, and all managers with ability less than q_H^* sort to some sector while all managers with ability greater than q_H^* sort to the other. In other words, in a threshold equilibrium (if one exists) the allocation sets Q_{L1}, Q_{L2}, Q_{H1}, and Q_{H2} all consist of single connected intervals. We wish to identify conditions under which such a simple sorting pattern emerges.

Consider first the allocation of workers. The following proposition provides a sufficient condition for the existence of an equilibrium in which all the most able workers sort to one sector and all the least able workers sort to the other.

Proposition 5 *Suppose that Assumption 2 holds and that*

$$\frac{\psi_{iL}(q_{H\min}, q_L)}{\gamma_i \psi_i(q_{H\min}, q_L)} > \frac{\psi_{jL}(q_{H\max}, q_L)}{\gamma_j \psi_j(q_{H\max}, q_L)}$$

for all $q_L \in S_L$, $i \neq j$, $i = 1,2$. *Then, in any competitive equilibrium with employment of workers in both sectors, the more able workers with* $q_L \geq q_L^*$ *are employed in sector* i *and the less able workers with* $q_H \leq q_H^*$ *are employed in sector* j, *for some* $q_L^* \in S_L$.

The proposition states that all high-ability workers—those with indexes above some threshold—will surely sort to sector i if the ratio of the elasticity of productivity with respect to worker ability to the elasticity of output with respect to the number of workers is higher in that sector when a given group of workers is teamed with the economy's *least able* manager than the similar elasticity ratio that applies for sector j when the workers instead are teamed there with the economy's *most able* manager. In such circumstances, the combinations of workers and managers that emerge in equilibrium cannot overturn the forces that we have previously identified that indicate sorting of the best workers to sector i.[21]

An analogous condition applies to the allocation of managers, namely

Proposition 6 *Suppose that Assumption 2 holds and that*

$$\frac{\psi_{iH}\left(q_H, q_{L\min}\right)}{\left(1-\gamma_i\right)\psi_i\left(q_H, q_{L\min}\right)} > \frac{\psi_{jH}\left(q_H, q_{L\max}\right)}{\left(1-\gamma_j\right)\psi_j\left(q_H, q_{L\max}\right)}$$

for all $q_H \in S_H$, $\quad i \neq j, \quad i = 1, 2$. *Then, in any competitive equilibrium with employment of managers in both sectors, the more able managers with* $q_H \geq q_H^*$ *are employed in sector* i *and the less able managers with* $q_H \leq q_H^*$ *are employed in sector* j, *for some* $q_H^* \in S_H$.

Here, the inequality ensures that the ranking of sectors by the elasticity ratio relevant for managers cannot be overturned even if we were to team a given manager with the economy's most able workers in sector j and with the economy's least able workers in sector i.

If the inequality in Proposition 5 holds for some i and j and that in Proposition 6 holds for some i' and j', then the allocations of the two factors generate a threshold equilibrium. Such an equilibrium can take one of two forms. First, we might have $i = i'$ and $j = j'$, in which case the most able workers and the most able managers will sort to the *same* sector. Alternatively, we might have $i = j'$ and $j = i'$, in which case the most able workers sort to the same sector as the least able managers, and *vice versa*. We refer to a sorting pattern that has the more able workers and managers employed in the same sector as an HH/LL equilibrium (for "high-high" and "low-low") and one that has the more able workers employed in the same sector as the less able managers as an HL/LH equilibrium (for "high-low" and "low-high").

It is easy to see that each of these types of equilibria can arise for certain productivity functions and parameter values. To illustrate this point, let us consider the limiting case in which $\psi_1(\cdot)$ and $\psi_2(\cdot)$ are only weakly log supermodular and, in particular, $\psi_i(q_H, q_L) = q_H^{\beta_i} q_L^{\alpha_i}$. In this case of Cobb-Douglas productivity (see footnote 17), the elasticity of productivity with respect to worker ability in sector i is a constant α_i and the elasticity of productivity with respect to manager ability in sector i is a constant β_i. In such circumstances, the elasticity ratios do not depend on the matches that form. Accordingly, an HH/LL equilibrium will arise if $\alpha_1/\gamma_1 > \alpha_2/\gamma_2$

[21] The strict log supermodularity of $\psi_i(\cdot)$ implies that $\psi_{iL}(q_H, q_L)/\psi_i(q_H, q_L)$ is increasing in q_H for every value of q_L. Therefore, if the inequality condition in the proposition holds, we must have

$$\frac{\psi_{iL}\left(q_H, q_L\right)}{\gamma_i\psi_i\left(q_H, q_L\right)} > \frac{\psi_{jL}\left(q_H', q_L\right)}{\gamma_j\psi_j\left(q_H', q_L\right)} \quad \text{for all } q_L \in S_L \text{ and all } q_H, q_H' \in S_H, \quad i \neq j.$$

Then, the ratio of elasticities for a given worker is greater in sector i than in sector j regardless of the matches that form in one sector or the other. In this case, the most able workers sort to the sector where the elasticity ratio is unambiguously highest.

and $\beta_1/(1-\gamma_1) > \beta_2/(1-\gamma_2)$, whereas an HL/LH equilibrium will arise if $\alpha_1/\gamma_1 > \alpha_2/\gamma_2$ and $\beta_2/(1-\gamma_2) > \beta_1/(1-\gamma_1)$. Indeed, in the Cobb-Douglas case—wherein the forces guiding the sorting of the two factors are of constant strength and independent of one another—a threshold equilibrium surely emerges. Qualitatively similar equilibria will exist for productivity functions that are strictly log supermodular if the departure from Cobb-Douglas is relatively slight and the ranges of ability levels for workers and managers are sufficiently small.

It is possible to provide a weaker sufficient condition for the existence of a threshold equilibrium of the HH/LL variety. If the most able managers sort to sector 1, this can only strengthen the incentives for the most able workers to sort there as well in the light of the complementarities between factors implied by log supermodularity. Similarly, if the most able workers sort to sector 1, the most able managers will have greater incentive to do likewise than otherwise. This reasoning motivates the following proposition, which we prove in the appendix.

Proposition 7 *Suppose that Assumption 2 holds. If*

$$\frac{\psi_{1H}(q_H,q_L)}{(1-\gamma_1)\,\psi_1(q_H,q_L)} > \frac{\psi_{2H}(q_H,q_L)}{(1-\gamma_2)\,\psi_2(q_H,q_L)} \quad \textit{for all } q_H \in S_H, \quad q_L \in S_L,$$

and

$$\frac{\psi_{1L}(q_H,q_L)}{\gamma_1\psi_1(q_H,q_L)} > \frac{\psi_{2L}(q_H,q_L)}{\gamma_2\psi_2(q_H,q_L)} \quad \textit{for all } q_H \in S_H, \quad q_L \in S_L,$$

then in any competitive equilibrium with employment of managers and workers in both sectors, the more able managers with $q_H \geq q_H^*$ are employed in sector 1 and the less able managers with $q_H \leq q_H^*$ are employed in sector 2, for some $q_H^* \in S_H$; the more able workers with $q_L \geq q_L^*$ are employed in sector 1 and the less able workers with $q_H \leq q_H^*$ are employed in sector 2, for some $q_L^* \in S_L$.

The difference in the antecedents in Propositions 5 and 6 on the one hand and in Proposition 7 on the other is that, in the former we compare the elasticity ratio for each factor when it is combined with the least able type of the other factor in one sector versus the most able type in the other sector, whereas in the latter we compare the elasticity ratios for common partners in the two sectors. The difference arises, because an HH/LL equilibrium has PAM within *and across* industries, whereas an HL/LH equilibrium has PAM only within industries. In an HL/LH equilibrium, an able manager in sector i might be tempted to move to sector j despite a generally greater responsiveness of productivity to ability in i, because the better workers have incentive to sort to j, and with log supermodularity of $\psi_j(\cdot)$, the able manager stands to gain most from this superior match. In contrast, in an HH/LL equilibrium, the able manager in sector i would find less able workers to match with were she to move to sector j, so the temptation to switch sectors in order to upgrade partners is not present.

We have provided sufficient conditions for the existence of a threshold equilibrium in which the allocation set for each factor and industry comprises a single, connected interval. These conditions are not necessary, however, because the matches available to types that are quite different from the

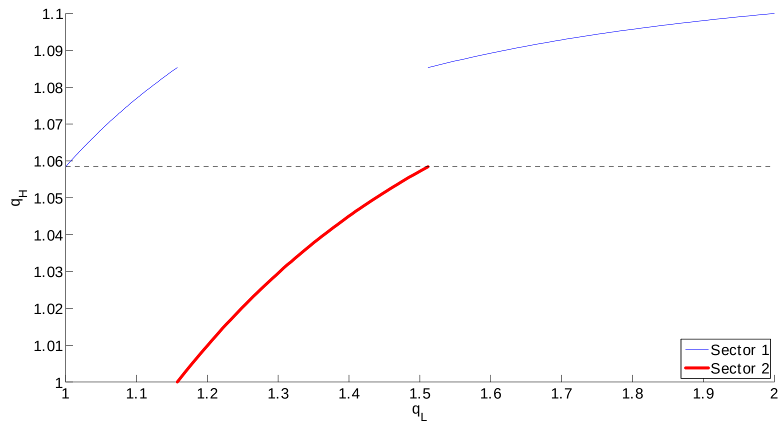

Figure 3: Matching: The most and least able workers and the most able mangers sort into sector 1

marginal type might not overturn their strong comparative advantage in one sector or the other. Nonetheless, not all parameter configurations give rise to equilibria with such a simple sorting pattern. An example of a more complex sorting pattern is illustrated in Figure 3.[22] The figure shows, for each worker type indicated along the horizontal axis, the sector in which that worker is employed and the type of the manager with whom he is matched. In this example, the most able and least able workers sort to sector 1 while an intermediate interval of worker types sort to sector 2. The firms in sector 1 hire the economy's most able managers whereas those in sector 2 hire those with ability below some threshold level. Notice that graphs M_1 and M_2 display the general properties that we described above; they are unions of connected sets, with a matching function $m(q_H)$ that is continuous and increasing within any such set. The figure reflects a "sorting reversal" for workers that arises because the elasticity ratio for labor is higher in sector 1 when worker ability is low or high, but higher in sector 2 for a middle range of abilities. Of course, other sorting patterns besides that depicted in Figure 3 also are possible.

Armed with an understanding of the forces that drive factor sorting, we will turn shortly to the relationship between factor endowments and trade and the effects of trade on the wage and salary distributions. But before that, it will prove helpful to examine how matching and factor prices are determined for some connected intervals of worker and manager types employed in a given sector.

4.2 Matching and Factor Prices Among a Group of Workers and Managers

Consider a subset of the factors employed and matched in some sector comprising the interval of managers $Q_H = [q_{Ha}, q_{Hb}]$ and the interval of workers $Q_L = [q_{La}, q_{Lb}]$.[23] Matching between these groups and all wages and salaries must satisfy the differential equations (17)-(19) for $q_H \in Q_H$ and $q_L = m(q_H) \in Q_L$, along with the zero-profit condition (16) and the boundary conditions,

[22] The functional forms and parameter values underlying this example are presented in Lim (2013).

[23] We omit for now the subscripts that identify the sector of employment, because we will be examining only this single group of workers and managers.

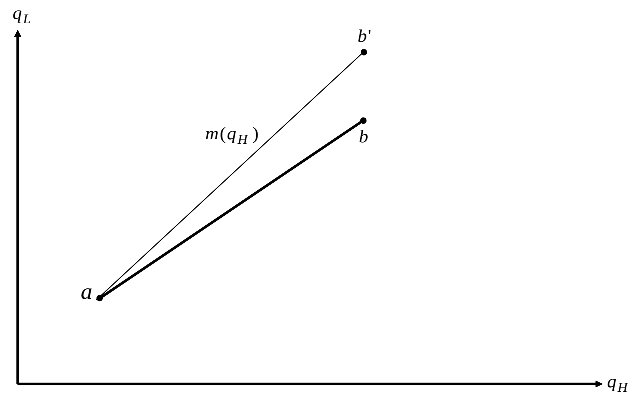

Figure 4: Shift in the matching function when q_L^b rises to $q_L^{b'}$

$q_{La} = m(q_{Ha})$ and $q_{Lb} = m(q_{Hb})$. The solution to this system, which is unique, is described in the appendix.

The solution has several notable properties. First, when the price of the good produced by these factors increases by some proportion, all wages for workers in Q_L and all salaries for managers in Q_H rise by this same proportion, while the matching between workers and managers remains as before. Second, when the ratio of the number of managers in Q_H to the number of workers in Q_L grows by some proportion $\hat{\eta}$, the wages of all workers in the group rise by the proportion $(1-\gamma)\hat{\eta}$, while the salaries of all managers in the group fall by the proportion $\gamma\hat{\eta}$. The change in relative numbers also has no effect on the matches that result.[24]

Now consider the effects of changes in the boundary points of Q_H and Q_L. Lemma 2 in the appendix establishes that, when (16)-(19) are satisfied for a given productivity function $\psi(\cdot)$ and for given parameters $p, \gamma, \bar{H}$ and $\bar{L}$ but with different boundary points, the corresponding matching functions can intersect at most once. Moreover, if such an intersection exists, the solution with the steeper matching function at the point of intersection also has lower wages and higher salaries for all ability levels that are common to the two settings; see Lemma 6 in the appendix. A steeper matching function means that managers are teamed with larger groups of workers, which implies a higher marginal product of the managerial input and a small marginal product of labor input at a given ability level for each factor.

Figure 4 illustrates how the matching function shifts when the uppermost boundary of the interval of workers rises from q_{Lb} to $q_{Lb'}$. Here, the matching functions that apply beforehand and afterward must intersect at the common boundary point, (q_{Ha}, q_{La}). By Lemma 2, we know that this can be the only intersection of the two curves, and then the fact that a manager with ability q_{Hb} initially matches with a team of workers with ability q_{Lb} but ultimately matches with those of ability $q_{Lb'}$ implies that the matching function shifts upward for all $q_H \in (q_{Ha}, q_{Hb}]$, as shown.

[24]See Lemma 1 in the appendix for a formal statement and proof of these results.

Finally, Lemma 6 implies that wages decline for all types in Q_L when additional workers are added to the upper end of the interval.

The re-matching depicted in Figure 4 has implications for within-group wage and salary inequality. Consider the wage distribution among workers in Q_L. The differential equation (17) implies that

$$\ln w_i\left(q_{Lc}\right) - \ln w_i\left(q_{Lc'}\right) = \int_{q_{Lc}}^{q_{Lc'}} \frac{\psi_{iL}\left[\mu\left(x\right), x\right]}{\gamma_i \psi_i\left[\mu\left(x\right), x\right]} dx, \quad \text{for all } q_{Lc,} q_{Lc'} \in Q_L\ , \tag{20}$$

where $\mu\left(\cdot\right)$ is the inverse of $m\left(\cdot\right)$.[25] Therefore, if all workers with ability levels between q_{Lc} and $q_{Lc'}$ are re-matched with managers that are less able than the ones they teamed with initially, the wage of the more able worker of type $q_{Lc'}$ will decline relative to that of the less able worker of type q_{Lc}. The downgrading of managers is detrimental to both of these workers, but especially so to the one with greater ability due to the complementarities between factor types. It follows that a re-matching of a group of workers with less able managers generates a narrowing of wage inequality within the group. By a similar argument (and using the differential equation (18) for salaries), the re-matching depicted in Figure 4 generates a spread in the salary distribution for managers in Q_H inasmuch as these managers all see their matches improve.[26]

Similar reasoning can be used to find the shift in the matching function—and the wage and salary responses—for changes in the other boundary points. For example, if the lower boundary of the interval of managers rises from q_{Ha} to $q_{Ha'}$, the matching function shifts downward (thereby connecting a point to the right of a in Figure 4 with point b), and thus the manager types that remain in the sector find that their matches deteriorate while all workers in Q_L match with better managers than before. Such a re-matching narrows the salary distribution while exacerbating wage inequality. In short, whenever the matches improve for a group of workers or managers working in some sector, the more able among them benefit the most and within-group inequality grows.

We are ready now to address the sources of comparative advantage and the impact of trade on wages and salaries.

4.3 Comparative Advantage

Consider two countries that have similar distributions of factor types but differ in their relative factor endowments. Suppose that a threshold equilibrium prevails in each country. Finally, suppose that the industries differ in their factor intensities. How do the relative output levels compare in the two countries?

Let us begin with the case of an HL/LH equilibrium in which the most able workers and least able managers are employed in sector 1. The solid lines in Figure 5 depict the qualitative features of the inverse matching function for country A in such circumstances. Notice that the equilibrium features PAM within sectors, but not across sectors, as we have previously described. Now compare

[25] If $\psi_i\left(\cdot\right)$ is strictly log supermodular, then $m\left(\cdot\right)$ is strictly increasing, and therefore must be invertible. If $\psi_i\left(\cdot\right)$ is only weakly log supermodular, we focus on the equilibrium with PAM (that surely exists), and then once again $m\left(\cdot\right)$ is invertible.

[26] Sampson (2012) derives a related result.

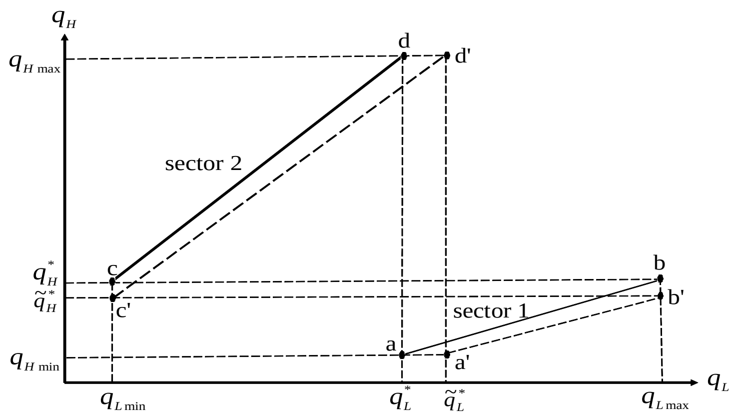

Figure 5: Sorting and matching: HL/LH equilibrium

country B, which we take to be the labor-abundant country; i.e., $\bar{H}^A/\bar{L}^A > \bar{H}^B/\bar{L}^B$. We prove in the appendix that the labor-abundant country allocates greater shares of both its managers and its workers to the labor-intensive industry. So, in country B, the worker threshold lies to the right of q_L^* and the manager threshold falls below q_H^* if and only if $\gamma_2 > \gamma_1$. In this case (depicted in the figure), the inverse matching function for country B must begin at a point such as c' (below c) and extend to a point such as d' (to the right of d). It cannot intersect cd twice (by Lemma 2 in the appendix), which means that it cannot intersect that curve at all. Therefore, a manager type that is employed in sector 2 in both countries achieves a match with more able workers in country B than in country A. For those types of workers employed in sector 2 in both countries, the matches with managers are better in country A. As for the matching among factors employed in sector 1, the curve for country B must begin at a point such as a' (to the right of a) and end at a point such as b' (below b). Here too the managers in country B are combined with more able workers than their counterparts of similar ability in country A, whereas workers of similar type are combined with less able managers in country B. Just the opposite is true about the relative positions of the matching functions and the comparisons of the match qualities when sector 2 is the more manager intensive; i.e., when $\gamma_1 > \gamma_2$. We prove in the appendix that, in either case, country A—with its relative abundance of managers—always exports the manager-intensive good.

Now consider an HH/LL equilibrium in which the most able workers and the most able managers sort to sector 1. The solid curve in Figure 6 depicts the inverse matching function for country A. It is continuous, monotonically increasing (PAM in each sector and economy-wide), and has a slope that rises at the threshold q_L^*. We show in the appendix that, in this case too, the labor-abundant country devotes a greater fraction of its managers and workers to the labor-intensive industry. Thus, if industry 2 is labor intensive ($\gamma_2 > \gamma_1$), the thresholds for country B must lie to the right of q_L^* and above q_H^*. The figure illustrates two different inverse matching functions that have this property. As is clear, one broken curve lies everywhere below the inverse matching

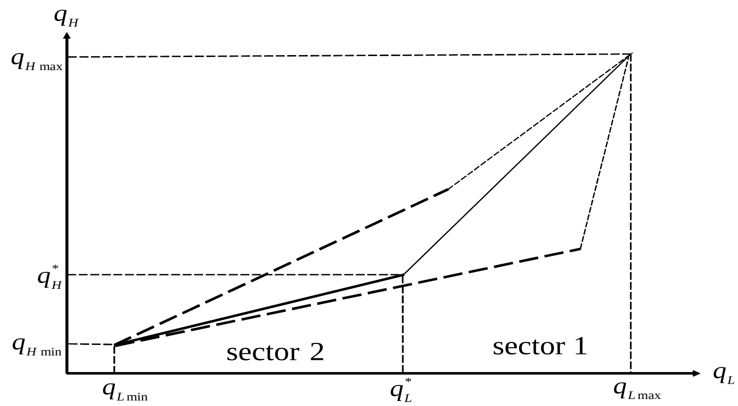

Figure 6: Sorting and matching: HH/LL equilibrium

function for country A, whereas the other lies everywhere above the solid curve. Therefore, it is not possible to say in general whether the workers or managers of a given ability level achieve better matches in the labor-abundant or in the manager-abundant country. Nonetheless, the fact that the labor-abundant country devotes a relatively greater share of its endowment of both factors to production in the labor-intensive industry suffices to ensure that this country produces relatively more of the labor-intensive good. In short, the Heckscher-Ohlin theorem extends to a setting with heterogeneous factors so long as a threshold equilibrium prevails in both countries and the countries share identical distributions of the factor types.[27]

Let us briefly consider trade between two countries that have similar aggregate factor endowments, similar distributions of manager types, but different distributions of heterogeneous workers. Suppose that the distribution of worker ability in country A is a rightward shift of that in country B; i.e., for each worker in country B, there is a counterpart in country A at the same place in the talent distribution that has $\lambda > 1$ times as much ability as measured by the index, q_L. In the appendix, we prove that, if $\psi_i(q_H, q_L) = q_H^{\beta_i} q_L^{\alpha_i}$ for $i = 1, 2$, then country A will have comparative advantage in producing good 1 if and only if $\alpha_1 > \alpha_2$. That is, the country with better workers exports the good produced in the industry in which productivity responds more elastically to worker ability. This result mirrors that for an economy with homogeneous managers, as reported in Proposition 3. We have not managed to prove an analogous analytical result for an economy with strictly log supermodular productivity functions. However, numerical simulations of the model—some of which are reported in Lim (2013)—support a similar conclusion in such circumstances.

[27] Recall that, in the case with Cobb-Douglas productivity, a threshold equilibrium must prevail in both countries. In the appendix we prove that, among countries with similar distributions of the two factors and Cobb-Douglas productivity in each industry, the labor-abundant country produces relatively more of the labor-intensive good.

4.4 The Effects of Trade on Wage and Salary Distributions

We turn now to the effects of trade on wages and salaries. In Section 3, where we studied an economy with homogeneous managers, we identified two considerations that color the link between output prices and factor prices. First, when trade causes the relative price of a country's export good to rise, the expansion of the export sector tends to benefit all types of the factor used intensively in that sector and to harm the factor used intensively in the import-competing sector. Second, when a factor is heterogeneous, trade tends to benefits those types of the factor that have comparative advantage in the export industry and to harm those types that have comparative advantage in the import-competing industry. Of course, these two influences on factor prices are familiar from the Heckscher-Ohlin economy and the Ricardo-Viner economy, respectively.

In an economy with two heterogeneous factors and complementarities between their abilities (or qualities), a new consideration comes into play. When productivity in each sector is a strictly log supermodular function of the employees' ability levels, the general equilibrium determines the matching of workers and managers within production units. Then, as output prices change and factors are re-allocated between sectors, the inflows of some marginal types into the expanding sector and the outflows of these types from the contracting sector causes a re-matching of types in each industry. This re-matching in turn affects each type's productivity and therefore the equilibrium rates of pay. We will find that re-matching introduces a mechanism by which trade alters within-industry wage and salary distributions.[28]

For concreteness, consider a country that exports good 2. In Figure 7, the solid curves cd and ab depict the country's (inverse) matching function prior to the opening of trade for the case of an HL/LH equilibrium in which the more able workers and less able managers sort to industry 1. To understand the distributional implications of trade in such a country, we examine the effects of an increase in the relative price of good 2. This draws workers and managers into sector 2, so that q_H^* falls and q_L^* rises.[29] The new boundary points are represented by c', d', a' and b'. As is evident from the figure, the new inverse matching function (represented by the broken curves) lies below the original function for all worker and manager types that remain in their original industry of employment. As a result, the opening of trade allows all managers except those that switch sectors to achieve better matches than before, while causing all workers except those that switch sectors to realize worse matches than before.

Proposition 8 summarizes these effects of trade on matching for the case of an HL/LH equi-

[28] We have also studied the effects of trade on factor prices in an economy with Cobb-Douglas productivity and report our findings in the appendix. As we have noted, the matching of managers and workers is not well determined in such an economy, since the relative productivity of two types of worker, for example, is not affected by the ability level of the manager with whom they might be matched. In such a setting, the rematching that results from trade is not determined, but neither is it material for factor prices. We find, with Cobb-Douglas productivity, that trade has no effect on within-industry wage or salary distribution, and the Stolper-Samuelson and Ricardo-Viner influences on factor prices are analagous to those in an economy with homogeneous managers.

[29] Before any factor reallocation, the increase in p_2 raises the value marginal product of the marginal workers and managers in sector 2 relative to those in sector 1. As factors reallocate, marginal products change and rematching occurs. But we show in the appendix that these secondary effects cannot overturn the impact effects, so that q_H^* must fall and q_L^* must rises in the setting described by the figure.

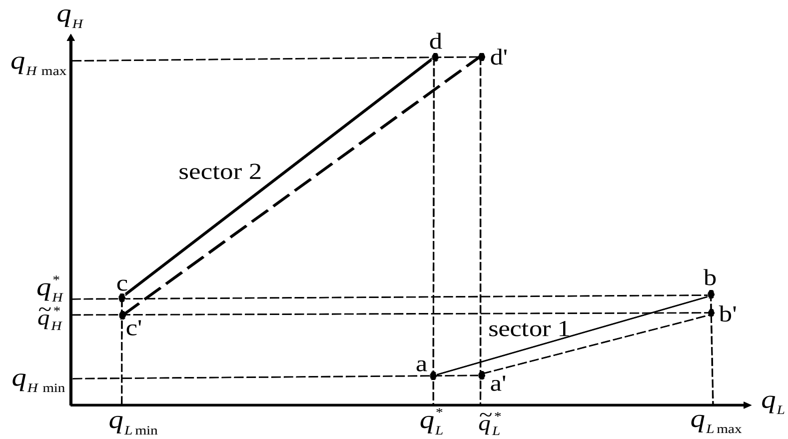

Figure 7: Effects of a rise in p_2 on matching: HL/LH equilibrium

librium and reports the implications for wage and salary inequality.

Proposition 8 *Suppose that: (i) Assumption 2 holds and (ii) the initial equilibrium is a threshold equilibrium with an HL/LH sorting pattern. Then an increase in p_2 (a) raises the labor cutoff q_L^* and reduces the manager cutoff q_H^* so that more workers and more managers are employed in sector 2; (b) worsens the matches for all workers except those that switch from sector 1 to sector 2; (c) improves the matches for all managers except those that switch from sector 1 to sector 2; (d) reduces within-industry wage inequality in both sectors and overall wage inequality in the economy; and (e) increases within-industry salary inequality in both sectors and overall salary inequality.*

Evidently, wage inequality falls among workers originally in industry 2 and among those remaining in industry 1. Take for example any two workers q_L^c and $q_L^{c'}$ such that $q_{L\min} \leq q_L^c < q_L^{c'} \leq q_L^*$. Both workers see their match deteriorate as a result of the increase in the price of good 2, but the re-matching harms the worker with ability $q_L^{c'}$ by relatively more due to the complementarities between factor types. This can be seen from (20), wherein the strict log supermodularity of $\psi(\cdot)$ implies that a downward shift in $\mu(\cdot)$ reduces the integrand on the right-hand side and thus reduces the relative wage of the more able worker in the pair. The same is true for any pair of workers with abilities between $\tilde{q}_L^*$ and $q_{L\max}$. Finally, consider a pair of workers that switch sectors; i.e., those that have ability levels between q_L^* and $\tilde{q}_L^*$. The relative wage of the less able worker in this pair must rise, because the elasticity of the wage schedule in (17) is determined after the price change by the elasticity ratio in sector 2, whereas before it was determined by the elasticity ratio in sector 1. Since the more able workers sort to sector 1, it must be that the former elasticity is smaller than the latter. It follows that wage inequality declines also among workers that switch sectors and therefore among all workers in the economy; see Figure 8 for an example.

What is the overall effect of the price change on the welfare of the various workers? There are

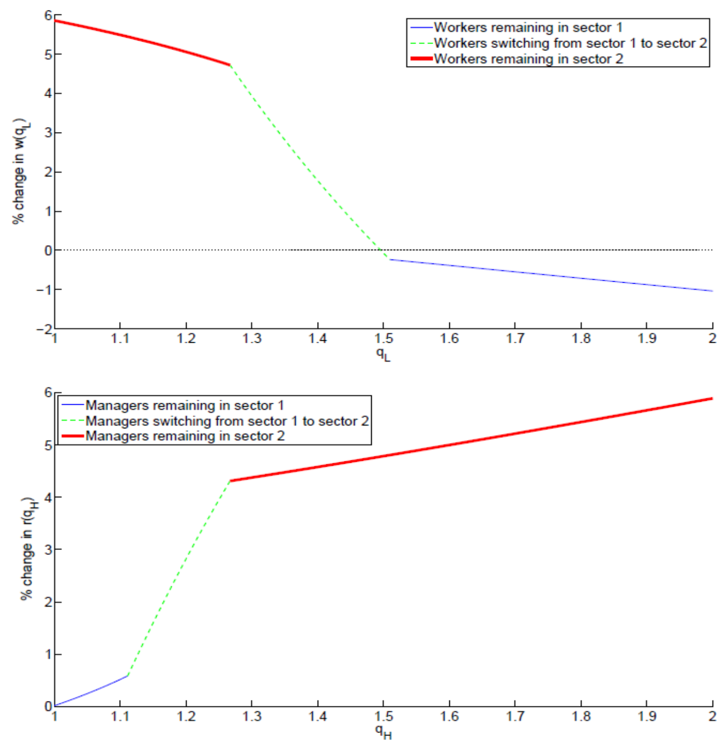

Figure 8: Effects of a 5% increase in p_2 on wages and salaries in an HL/LH equilibrium

several possibilities that can emerge, as can be seen in the numerical simulations presented by Lim (2013). First, if sector 1 is labor intensive and the difference in factor intensities across sectors is large relative to the specificity of the heterogeneous factors, then the Stolper-Samuelson forces dominate. In such circumstances, real wages decline for all workers while real salaries increase for all managers. Of course, if sector 2 is the labor-intensive industry, then the opposite outcomes are possible, with real gains for all workers and losses for all managers.

Figure 8 depicts the wage and salary responses for a less extreme case.[30] Here, sector 2 is labor intensive and p_2 rises by 5%. All workers initially in sector 2 see their wages rise and those at the bottom end of the ability distribution enjoy a wage hike in excess of 5%. Meanwhile, the workers who remain in sector 1 suffer a decline in wages despite the rise in the price of the labor-intensive good. These workers suffer from their comparative disadvantage in the expanding sector. As for managers, those at the top end of the ability distribution gain the most and some see salary improvements in excess of 5%. Those at the bottom of the ability distribution enjoy welfare gains only if they devote little of their income to the export good. The figure shows the widening of salary inequality among managers.

A host of other possible configurations can emerge, but all can be understood similarly with reference to the relevant factor intensities and sector specificities; see Lim (2013) for examples. Rather than dwell on these cases, we turn now to the wage and salary effects of trade in an

[30] See Lim (2013) for the parameter values and functional forms that underlie this figure.

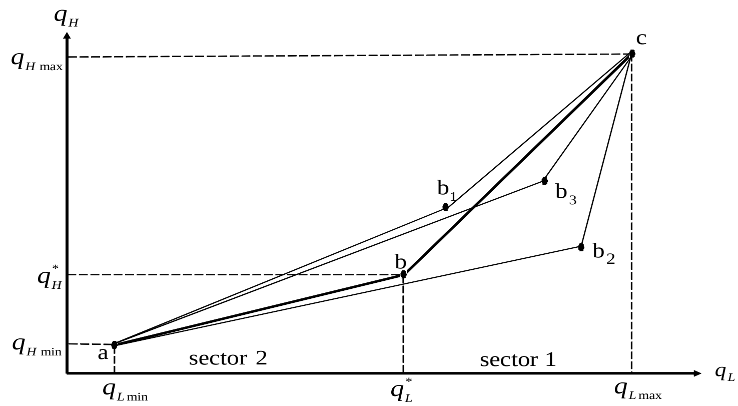

Figure 9: Impact of a rise in p_2 on matching: HH/LL equilibrium

HH/LL equilibrium. Recall the matching and sorting patterns for such an equilibrium that were displayed in Figure 6. We show in the appendix that, when the price of good 2 rises in such a setting, sector 2 expands by attracting both additional workers and additional managers. It follows that both q_L^* and q_H^* rise. In this case, the implications for matching vary according to whether the movement of workers or the movement of managers dominates.

Figure 9 illustrates the various possibilities.[31] The thick curve abc represents the initial inverse matching function. Now suppose that q_L^* rises only modestly, while q_H^* rises more dramatically.[32] Then the new equilibrium would be represented by an inverse matching function such as ab_1c. In the event, all workers' matches improve following the price hike, whereas all managers see their matches deteriorate. Alternatively, the inflow of workers to sector 2 can be large relative to that for managers, in which case q_L^* could expand greatly compared to the expansion in q_H^*. This possibility is illustrated by the inverse matching function ab_2c in the figure, and it implies a deterioration in match quality for all workers and an improvement for all managers. Finally, the inverse matching function ab_3c depicts an intermediate case. Notice that the matches improve for all workers initially in sector 2 but deteriorate for all those remaining in sector 1.

Let us focus on the case where the outcome is an inverse matching function such as ab_1c to discuss the implied wage and salary responses. Since workers' matches improve, wages rise faster with ability than before. Since managers' matches deteriorate, the opposite is true of managerial salaries. Notice that the inverse matching function has a steeper slope at point a in the new equilibrium than before the price change. It follows from Lemma 6 that the wage of the least able workers must rise. These workers benefit directly from the increase in p_2 and indirectly from the improvement in their matches. The direct benefit alone matches the proportional increase in price, so these workers enjoy real income gains. At the opposite end of the spectrum, the most able

[31] Lim (2013) provides numerical examples of each along with the underlying parameter values.

[32] This outcome plausibly arises when sector 2 is considerably more manager intensive than sector 1.

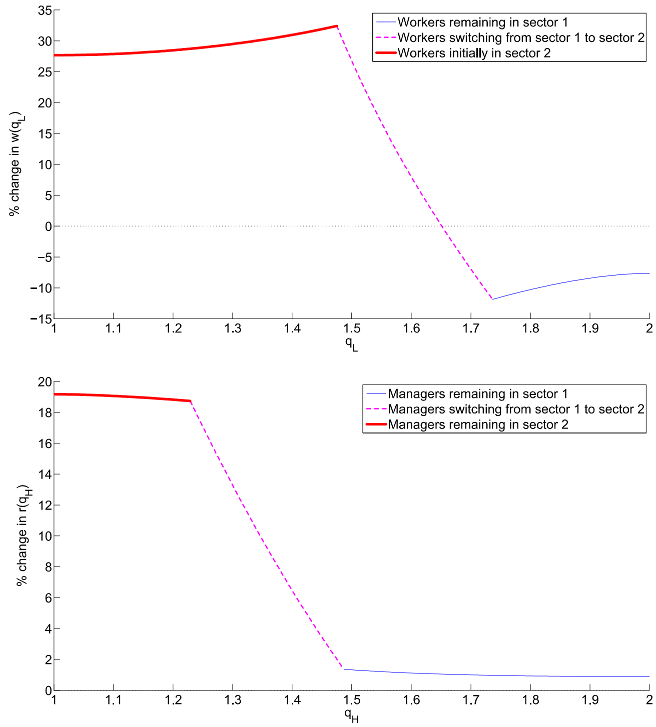

Figure 10: Effects of a 20% increase in p_2 on wages and salaries in an HH/LL equilibrium

workers must lose. The change in p_2 has no direct effect on their value marginal product. Since the new inverse matching function is flatter at point c than the initial function, Lemma 6 implies that these workers suffer a decline in nominal wages. The gain in real income for the least able workers and the loss for the most able workers represents a narrowing of wage inequality across sectors, whereas the improved matching implies that wages are more unequal within each sector.

Figure 10 presents another example drawn from Lim (2013). Notice that the least able workers enjoy real income gains, though not as large as for those more able than themselves who initially are employed in the same sector. Meanwhile, the most able workers lose, but not as much as those less able than themselves who remain in sector 1. The figure also shows the effect on managerial salaries. In this example, all managers realize income gains in terms of good 2 but losses in terms of good 1. These gains are smaller and the losses larger as we move up the salary distribution. A decline in $r\left(q_{H\,\min}\right)/p_2$ is guaranteed in this case, because the direct effect for the least able managers is a salary increase proportional to the rise in p_2, but the steepening of the inverse matching function at a implies that their salaries must fall relative to the price of what they produce. The rise in $r\left(q_{H\,\max}\right)/p_1$ also is guaranteed, because the inverse matching function is flatter at point c than before. Finally, we know that the new salary function is flatter than the old both for managers initially in sector 2 and for those that remain in sector 1, because the deterioration in match quality hits especially hard for the more able managers in any sector.

If the inverse matching function instead is qualitatively like that depicted by ab_2c in Figure 9, then the outcomes are just the opposite. Low-ability managers gain from an increase in p_2, because their value marginal product rises in proportion to the price hike and rises further as a result of the re-matching. High-ability managers lose in real terms, because $r\left(q_{H\max}\right)/p_1$ falls. All wages rise, albeit less than in proportion to the price increase. The wage hikes are proportionally greatest for those at the bottom end of the ability distribution. As a result of these factor price responses, wage inequality declines both within and between sectors, whereas salaries become more unequal within sectors, but those at the bottom who are employed in sector 2 gain relative to those at the top who are employed in sector 1.

Finally, if the inverse matching function is like that depicted by ab_3c, then the outcomes are a mix of those described above. In this case, all workers initially employed in sector 2 must benefit from the price increase, while all managers initially employed in sector 1 must lose. The low-ability managers and the high-ability workers both gain in compensation relative to the price of good 1, but lose relative to the price of good 2. Lim (2013) provides numerical examples.

5 Labor Market Frictions

Until now, we have assumed that labor markets flawlessly and costlessly allocate the various types of labor to their most efficient uses. Of course, the smooth functioning of labor markets is notoriously suspect and worker heterogeneity would only seem to exacerbate the potential difficulties. In this section, we show how a simple form of search frictions can be incorporated into the analysis. The extension allows us to discuss the distribution of unemployment rates across the ability spectrum alongside the distribution of wages.

To keep matters simple, we continue to assume a frictionless market for managers. In other words, firms can hire managers of whatever ability and in whatever numbers they wish by offering a competitive salary.[33] But firms must search for their workers and workers for jobs. We follow Peters (1991, 2000), Acemoglu and Shimer (1999), Burdett et al. (2001), Eeckhout and Kircher (2010a, 2012), and others in modeling labor-market frictions with "directed search," whereby firms post costly "vacancies" that announce their compensation offers and targeted workers and employee-seeking firms meet randomly. In particular, we extend the approach of Eeckhout and Kircher that allows for worker heterogeneity and multiple hires per firm to an environment with two industries.

Suppose, as before, that the output in industry i of a production unit comprising a manager of ability q_H and ℓ workers of ability q_L is given by (1). A firm (or entrepreneurial manager) hires workers by posting vacancies. Each posting costs c_i units of the the firm's final output. A posting lists the ability level q_L that the firm targets and the wage ω that it will pay to any employee of this type. We assume that the firm can commit to these job attributes, in the sense that it will not hire workers with ability different from the posted level nor attempt to renegotiate its wage offer

[33] Perhaps the best way to justify this assumption is to imagine the manager as an entrepreneur, as in Lucas (1978). Then it is the manager that searches for employees and her salary amounts to the residual profits after wages and hiring costs are paid. Alternatively, one might think of the second factor as being *capital*, instead of managers, in which case an assumption that firms can readily find machines of the quality they desire is not so hard to swallow.

after it meets with a job applicant.[34] The firm chooses v, the number of its vacancies, to maximize profits.

Workers are risk neutral. Each worker applies for a single job of his choosing.[35] Workers consider only the jobs for which they are qualified, because firms will not hire types different from those targeted in their announcements. Among relevant jobs, each worker applies for the position that offers the greatest expected income. In equilibrium, workers must be indifferent among the range of openings posted for their type.

Let s be the number of workers seeking jobs at a firm that has posted v vacancies. We assume that the search process results in the consummation of $M(s,v)$ jobs, where

$$M(s,v) = Bs^{\tau}v^{1-\tau}, \tag{21}$$

$B > 0$ and $0 < \tau < 1$.[36] For a firm, the probability of filling any given vacancy is $\delta_v(s/v) = B(s/v)^{\tau}$, whereas for a worker the probability of a successful application is $\delta_s(s/v) = B(s/v)^{-(1-\tau)}$. The former is increasing in s/v, while the latter is decreasing in s/v; i.e., a firm's chances of filling a vacancy improve and a worker's chances of landing a job decline with the number of applicants per posting.

Now let $w(q_L)$ be the *expected wage* that workers of type q_L obtain in equilibrium, which each firm takes as given. A firm must offer at least this expected wage or it will find itself without applicants; and it has no reason to offer more. In equilibrium, a firm with v vacancies that offers a wage ω targeted to workers with ability q_L attracts s applicants, where s is such as to make the applicants indifferent between the firm's openings and their other opportunities; i.e., s solves $\delta_s(s/v)\omega = w(q_L)$. Using (21), this can be rewritten as

$$\frac{s}{v} = \left[\frac{B\omega}{w(q_L)}\right]^{\frac{1}{1-\tau}}. \tag{22}$$

Equation (22) is the main building block in a model with directed search; it ties the wage announcement ω to the endogenous number of applications per vacancy s/v, which in turn determines the firm's fill rate, $\delta_v(s/v)$.[37] Given the expected wage $w(q_L)$, the firm can use (22) to compute

[34] Alternatively, we could allow a firm to post a wage schedule and to hire any worker it happens to meet at the wage specified by the schedule. If each vacancy generates at most one meeting with a job applicant, then it is never optimal for the firm to induce applications from more than one type of worker; see Eeckhout and Kircher (2010a, 2010b) for proof of this assertion in related environments. In such circumstances, there is no loss of generality in assuming that the firm targets only one type of worker. Shimer (2005) studies a setting in which one vacancy can result in multiple meetings with potential employees. Then, in the general, it is optimal for any firm to induce applications from several different types. We do not explore this possibility here.

[35] This assumption is common in the literature on direct search. Galenianos and Kircher (2009) describe settings in which the restriction to one application per worker does not change the qualitative predictions of the model.

[36] The job-search literature refers to $M(s,v)$ as a "matching function" but we eschew that terminology so as to avoid confusion with the function that "matches" workers and managers, $q_L = m(q_H)$. The Cobb-Douglas form for $M(\cdot)$ is common in the literature, and is implicitly coupled with the usual restriction that B is sufficiently small to imply meeting probabilities below unity for both vacancies and workers.

[37] Peters (1991, 2000) and Burdett et al. (2001) provide microfoundations for a relationship similar to (22). They begin by assuming a finite number of jobs and vacancies and then allow the economy to grow large without bound.

the number of workers that will seek its employment and thus the number of workers $\ell = M(s, v)$ that it will succeed in hiring. Again using (21), together with (22), we see that a firm that posts v vacancies targeted at workers with ability q_L and that offers a wage of ω manages to hire ℓ workers, where

$$\ell = B^{\frac{1}{1-\tau}} \left[\frac{\omega}{w(q_L)}\right]^{\frac{\tau}{1-\tau}} v. \tag{23}$$

Evidently, hires are proportional to the number of vacancies and rise with the ratio of the firm's wage offer to the workers' outside option.

Now consider the profit-maximization problem facing a firm with a manager of ability q_H that chooses to operate in industry i. The firm pays $p_i c_i v$ to post v vacancies and pays ω to each of the ℓ workers that it eventually hires. Its profits are given by

$$\pi_i = p_i \psi_i(q_H, q_L)\ell^{\gamma_i} - \omega\ell - p_i c_i v - r(q_H),$$

where $r(q_H)$ as before represents the manager's salary. Then, using (22) and the first-order condition for the firm's optimal choice of wage offer, we can re-express its profits as

$$\pi_i = p_i \varphi_i(q_H, q_L)s^{\zeta_i} - w(q_L)s - r(q_H),$$

where

$$\varphi_i(q_H, q_L) \equiv [1-(1-\tau)\gamma_i]\left[\frac{(1-\tau)\gamma_i}{c_i}\right]^{\frac{(1-\tau)\gamma_i}{1-(1-\tau)\gamma_i}} B^{\frac{\gamma_i}{1-(1-\tau)\gamma_i}} \psi_i(q_H, q_L)^{\frac{1}{1-(1-\tau)\gamma_i}}$$

and

$$0 < \zeta_i \equiv \frac{\tau\gamma_i}{1-(1-\tau)\gamma_i} < 1.$$

Notice that this expression for profits has the same mathematical properties as the profit function $\pi_i = p_i\psi_i(q_H, q_L)\ell^{\gamma_i} - w(q_L)\ell - r(q_H)$ that we encountered in Section 4, because if $\psi_i(q_H, q_L)$ satisfies part (ii) of Assumption 3 (i.e., it is strictly increasing, continuously differentiable, and strictly log supermodular) so too does $\varphi_i(q_H, q_L)$, and ζ_i like γ_i is between zero and one.[38] In other words, the firm's choice about the number of job applications to invite in a setting with search frictions is much like its choice about the number of workers to hire in a setting without them. The first-order condition for s implies

$$s = \left[\frac{\delta_i p_i \varphi_i(q_H, q_L)}{w(q_L)}\right]^{\frac{1}{1-\delta_i}}, \tag{24}$$

This generates a balls–and-urns type function for applicants per vacancy, rather than the Cobb-Douglas form that is more commonly assumed. Galenianos and Kircher (2012) extends their setup to generate CES and Cobb-Douglas matching functions. With but a few exceptions, the literature on directed search specifies the matching function individually for each vacancy, and we follow in this tradition.

[38] Note too that if $\psi_i(q_H, q_L)$ is a product of power functions, so too is $\varphi_i(q_H, q_L)$. And if $\psi_i(q_H, q_L)$ has a constant elasticity of substitution between q_H and q_L, so too does $\varphi_i(q_H, q_L)$.

which generates the profit function

$$\pi_i(q_H, q_L) = \bar{\zeta}_i p_i^{\frac{1}{1-\delta_i}} \varphi_i(q_H, q_L)^{\frac{1}{1-\zeta_i}} w(q_L)^{-\frac{\zeta_i}{1-\zeta_i}} - r(q_H),$$

where $\bar{\zeta}_i \equiv \zeta_i^{\frac{\zeta_i}{1-\zeta_i}}(1-\zeta_i)$. This expression has much the same form as (14), which applies in the absence of search frictions. Finally, the analog to the labor-market clearing condition from before is the requirement that the aggregate number of applications induced by firms operating in industry i and targeting workers of ability q_L must equal the number of workers with that ability level that sort to the sector in search of a job. With these observations, we conclude that the equilibrium expected wage function $w(q_L)$, salary function $r(q_H)$ and matching function $q_L = m(q_H)$ can be characterized as the solution to three differential equations analogous to (17)-(19), a zero profit condition analogous to (16), and a set of boundary conditions. Evidently, comparative advantage again derives from a country's relative factor endowments and its distributions of worker and manager ability. Moreover, since $\zeta_1 > \zeta_2$ if and only if $\gamma_1 > \gamma_2$, the cross-sectoral differences in factor intensities interact with differences in factor endowments to determine the pattern of trade in much the same way as before. The search frictions themselves are not an independent source of comparative advantage so long as these frictions are similar in the two sectors.[39]

The model with search frictions features different employment rates across the range of ability levels. In order to discuss the impact of trade on employment, we combine the optimal choice of wage offer with a firm's desired number of applications per manager to derive

$$\omega(q_L) = B^{-\gamma_i} \left(\frac{1-\tau}{\tau p_i c_i}\right)^{-(1-\tau)\gamma_i} w(q_L)^{1-(1-\tau)\gamma_i}.$$

The expected wage $w(q_L)$ must be an increasing function of ability. It follows that, among workers that seek employment in a given industry i, those with greater ability see higher posted wages for the jobs they pursue. Next, we substitute this expression for $\omega(q_L)$ into (23) to derive an expression for the employment rate for workers of ability q_L, namely

$$\frac{\ell}{s} = B^{\gamma_i} \left(\frac{1-\tau}{\tau c_i}\right)^{(1-\tau)\gamma_i} \left[\frac{w(q_L)}{p_i}\right]^{(1-\tau)\gamma_i}. \tag{25}$$

Since the expected wage on the right-hand side is an increasing function of ability, we conclude that so too is the employment rate among workers seeking jobs in a given industry. We record our findings in

Proposition 9 *Suppose that Assumption 2 holds. Let $q_L', q_L'' \in Q_i$, with $q_L' > q_L''$. Then the job listings targeted to workers with ability q_L' offer a higher expected wage and a greater probability of employment than those targeted to q_L''. The opening of trade causes the within-sector inequality of expected wages and employment rates to move in the same direction.*

[39] If the number of meetings in (21) varies by sector, then it is immediate from the definition $\zeta_i \equiv \tau_i\gamma_i/(1-(1-\tau_i)\gamma_i)$ that the search process constitutes an additional source of comparative advantage.

In a setting with search frictions, the opening of trade affects differently the employment rates at different ability levels. Let us consider just one example to illustrate how the analysis can be performed. Suppose a country has an HL/LH sorting pattern such as that depicted in Figure 5 and that the country exports good 2. The opening of trade generates an increase in p_2. Figure 7 shows the effects of such a price change on the matching of worker and manager types in each sector. As we have seen, the workers who do not switch sectors find themselves teamed with a less able manager than before. Now, Figure 8 can be interpreted as illustrating the predicted impact on *expected* wages. The figure shows an increase in $w(q_L)/p_2$ for some of the least able workers, who sort to sector 2, a decline in $w(q_L)/p_2$ for some moderately able workers that sort to sector 2, and a decline in $w(q_L)/p_1$ for the most able workers, who sort to sector 1.

We refer now to equation (25), which applies in the presence of search frictions. The equation implies that the employment rate rises for the aforementioned group of least able workers while it falls for those with moderate and high ability. Overall, the distribution of employment rates becomes more equal across the worker population. Of course, the effects of trade on the distribution of employment would be just the opposite if the country instead imported good 2. Evidently, trade can widen or narrow the inequality in employment rates across the ability distribution according to the sorting pattern that is realized and the comparative advantage of the country. The determinants of these outcomes in an economy with directed search are similar to the determinants of wage inequality in an economy that has frictionless labor markets.

6 Concluding Remarks

In this paper, we have extended the familiar two-sector, two-factor model of international trade to include heterogeneous factors of production. In a model with factor heterogeneity, we can examine the determinants of factor sorting to industries and the determinants of factor matching within industries. When the productivity of a production unit depends on both the manager's and workers' abilities—and particularly when there are strong complementarities between the two—the forces that guide sorting and matching become inextricably linked. The economy-wide pattern of factor assignments can be subtle and complex even in the presence of strong complementarities that dictate positive assortative matching within every sector.

A model with heterogeneous factors allows a more complete analysis of the distributional effects of trade than is possible in one with homogeneous factors. In particular, we can ask how the opening of trade or trade liberalization affects the wage and salary distributions over the entire range of compensation levels. In general, there are three considerations that determine the effects of trade on the income of a particular individual. First, as in the standard Heckscher-Ohlin world with homogeneous factors, there is the question of whether the export sector is intensive in the use of workers or managers. Second, as in the standard Ricardo-Viner world with factor specificity, there is the question of whether an individual's type generates a personal comparative advantage in the export sector or the import-competing sector. Finally, and most novel, there is the question of how trade affects the individual's match with other factors of production. If a change in trade conditions causes a worker to re-match with a better manager than before, then his productivity will improve and his wage will receive an upward boost. If instead a worker's match deteriorates, then his wage may suffer. Interestingly, the effects of trade on wage or salary inequality across sectors may run counter to the effects on inequality within a sector.

We have shown that the Heckscher-Ohlin theorem extends to a setting with heterogeneous factors provided that the countries share similar distributions of worker and managerial talent. But we have also noted how differences in the distributions of talent can be an independent source of comparative advantage. A country that has more able workers than another—in the sense of a rightward shift in the talent distribution—will produce relatively more of the good for which productivity responds more elastically to ability.

Finally, we have incorporated search frictions. In a simple setting with directed search, firms create vacancies and make wage offers to workers of a targeted type. In such a setting, trade affects not only the distribution of wages but also the distribution of employment rates across the different types of workers. We provide an example in which the main insights from the earlier analysis carry over without modification to an environment with unemployment. But much work remains to the elucidate the connection between trade and the efficiency of matching and to understand how globalization affects equilibrium unemployment rates for different types of workers.

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Appendix for "Matching and Sorting in a Global Economy"

by

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Appendix A

This appendix provides proofs of results stated in the main text.

Proofs for Section 3

First, note that in the system comprising (6)-(9), a proportional increase in the number of managers and workers, $\bar{H}$ and $\bar{L}$, raises H_1 by the same factor of proportionality and leaves the marginal worker q_L^* and the wage anchors w_1 and w_2 unchanged. Therefore, the outputs of the two goods rise equiproportionately, so that the ratio X_1/X_2 does not change. Accordingly, to find the impact of $\bar{H}/\bar{L}$ on X_1/X_2 it suffices to examine the effects of a change in one factor, say $\bar{L}$.

Differentiating the equilibrium system (6)-(9), we obtain

$$\begin{pmatrix} 1 & -1 & s_L\left(q_L^*\right) & 0 \\ -\frac{\gamma_1}{1-\gamma_1} & \frac{\gamma_2}{1-\gamma_2} & 0 & 0 \\ 0 & \frac{E_2}{1-\gamma_2} & \bar{L}\tilde{\psi}_2\left(q_L^*\right)^{1/\gamma_2}\phi_L\left(q_L^*\right)q_L^* & E_2H_1/H_2 \\ \frac{E_1}{1-\gamma_1} & 0 & -\bar{L}\tilde{\psi}_1\left(q_L^*\right)^{1/\gamma_1}\phi_L\left(q_L^*\right)q_L^* & -E_1 \end{pmatrix}\begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{q}_L^* \\ \hat{H}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -E_2 \\ -E_1 \end{pmatrix}\widehat{\bar{L}} + \begin{pmatrix} 0 \\ -\frac{1}{1-\gamma_1} \\ 0 \\ \frac{E_1}{1-\gamma_1} \end{pmatrix}\hat{p}_1, \tag{26}$$

where$E_i = H_i\left(\frac{\gamma_i p_i}{w_i}\right)^{\frac{1}{1-\gamma_i}}$, $i = 1,2;$ $H_2 = \bar{H} - H_1$.

Let D_{ho} be the determinant of the matrix on the left-hand side of (26). Then

$$D_{ho} = \bar{L}\phi_L\left(q_L^*\right)q_L^*\frac{w\left(q_L^*\right)}{w_1w_2}H_1\frac{\left(\gamma_2-\gamma_1\right)^2}{\left(1-\gamma_1\right)^2\left(1-\gamma_2\right)^2} + \frac{s_L\left(q_L^*\right)}{\left(1-\gamma_1\right)\left(1-\gamma_2\right)}\left(\gamma_1+\gamma_2\frac{H_1}{H_2}\right)E_1E_2,$$

because (8), (9) and the definition of E_i imply that

$$\begin{aligned} \tilde{\psi}_1\left(q_L^*\right)^{1/\gamma_1}E_2\frac{H_1}{H_2} - \tilde{\psi}_2\left(q_L^*\right)^{1/\gamma_2}E_1 &= H_1\left[\tilde{\psi}_1\left(q_L^*\right)^{1/\gamma_1}\left(\frac{\gamma_2p_2}{w_2}\right)^{\frac{1}{1-\gamma_2}} - \tilde{\psi}_2\left(q_L^*\right)^{1/\gamma_2}\left(\frac{\gamma_1p_1}{w_1}\right)^{\frac{1}{1-\gamma_1}}\right] \\ &= \frac{w\left(q_L^*\right)}{w_1w_2}H_1\frac{\gamma_2-\gamma_1}{\left(1-\gamma_1\right)\left(1-\gamma_2\right)}. \end{aligned} \tag{27}$$

It follows that $s_L\left(q_L^*\right) > 0 \Rightarrow D_{ho} > 0$.

We now use (26) to calculate the response of H_1 to an increase in labor supply $\bar{L}$, which yields

$$\hat{H}_1D_{ho} = \frac{\gamma_1-\gamma_2}{\left(1-\gamma_1\right)\left(1-\gamma_2\right)}\left[E_1\bar{L}\tilde{\psi}_2\left(q_L^*\right)^{1/\gamma_2}\phi_L\left(q_L^*\right)q_L^* + E_2\bar{L}\tilde{\psi}_1\left(q_L^*\right)^{1/\gamma_1}\phi_L\left(q_L^*\right)q_L^* + E_1E_2s_L\left(q_L^*\right)\right]\widehat{\bar{L}}.$$

Therefore, given $s_L\left(q_L^*\right) > 0$, an increase in $\bar{L}$ raises the number of managers in sector 1 if and only if $\gamma_1 > \gamma_2$. When H_1 increases, X_1/X_2 does so as well. It follows that the country with relatively more workers produces relatively more of the labor-intensive good. This proves Proposition 2.

Next we calculate the response of the two wage anchors to changes in the price of good 1. From (26) and (27), we obtain

$$\hat{w}_1 D_{ho} = \bar{L}\phi_L\left(q_L^*\right)q_L^*\frac{1}{1-\gamma_1}\frac{w\left(q_L^*\right)}{w_1 w_2}H_1\frac{\gamma_1-\gamma_2}{\left(1-\gamma_1\right)\left(1-\gamma_2\right)}\hat{p}_1 + s_L\left(q_L^*\right)\frac{E_1 E_2}{\left(1-\gamma_1\right)\left(1-\gamma_2\right)}\left(1+\gamma_2\frac{H_1}{H_2}\right)\hat{p}_1,$$

$$\hat{w}_2 D_{ho} = \bar{L}\phi_L\left(q_L^*\right)q_L^*\frac{1}{1-\gamma_1}\frac{w\left(q_L^*\right)}{w_1 w_2}H_1\frac{\gamma_1-\gamma_2}{\left(1-\gamma_1\right)\left(1-\gamma_2\right)}\hat{p}_1 - s_L\left(q_L^*\right)\frac{E_1 E_2}{1-\gamma_1}\frac{H_1}{H_2}\hat{p}_1.$$

It follows that $s_L\left(q_L^*\right) > 0 \Rightarrow \left(\hat{w}_1 - \hat{w}_2\right)/\hat{p}_1 > 0$, which implies that $\hat{w}_1 > \hat{w}_2$ when $\hat{p}_1 > 0$, as stated in part (i) of Proposition 4. We also calculate the response of the managers' salary, using (10) with $i = 2$. We find

$$\hat{r} = -\frac{\gamma_2}{1-\gamma_2}\hat{w}_2.$$

Evidently, the managers' salary moves in the opposite direction to the wage anchor in sector 2.

Now consider the cases discussed in parts (ii)-(iv) of Proposition 4. In case (ii) we have $\gamma_1 \approx \gamma_2$ and therefore

$$\hat{w}_1 \approx \frac{\gamma_1^{-1}+\frac{H_1}{H_2}}{1+\frac{H_1}{H_2}}\hat{p}_1,$$

$$\hat{w}_2 \approx -\frac{\left(1-\gamma_1\right)\frac{H_1}{H_2}}{\gamma_1\left(1+\frac{H_1}{H_2}\right)}\hat{p}_1,$$

$$\hat{r} \approx \frac{\frac{H_1}{H_2}}{1+\frac{H_1}{H_2}}\hat{p}_1.$$

It follows that $\hat{w}_1 > \hat{p}_1 > \hat{r} > 0 > \hat{w}_2$, which proves part (ii) of the proposition. In cases (iii) and (iv), we have $s_L\left(q_L^*\right) \approx 0$, which implies

$$\hat{w}_1 \approx \hat{w}_2 \approx \frac{1-\gamma_2}{\gamma_1-\gamma_2}\hat{p}_1,$$

$$\hat{r} = -\frac{\gamma_2}{\gamma_1-\gamma_2}\hat{p}_1.$$

Therefore, if $\gamma_1 > \gamma_2$ then $\hat{w}_1 \approx \hat{w}_2 > \hat{p}_1 > 0 > \hat{r}$ and if $\gamma_1 < \gamma_2$ then $\hat{r} > \hat{p}_1 > 0 > \hat{w}_1 \approx \hat{w}_2$, which proves parts (iii) and (iv).

We now consider the impact of a rightward shift of the density function $\phi\left(q_L\right)$, as defined in (12). To perform these comparative statics, we can equivalently hold the distribution of types constant but endow a worker of type q_L with λq_L units of ability. In each sector, the demand for efficiency units of labor must equal the supply. A worker in sector i of type q_L provides $\tilde{\psi}_i\left(\lambda q_L\right)^{1/\gamma_i}$ units of efficiency labor. The labor-market clearing conditions should now be written as

$$H_1\left(\frac{\gamma_1 p_1}{w_1}\right)^{\frac{1}{1-\gamma_1}} = \bar{L}\int_{q_L^*}^{q_{L\max}}\tilde{\psi}_1(\lambda q)^{1/\gamma_1}\phi_L\left(q\right)dq \tag{28}$$

and

$$\left(\bar{H}-H_1\right)\left(\frac{\gamma_2 p_2}{w_2}\right)^{\frac{1}{1-\gamma_2}} = \bar{L}\int_{q_{L\min}}^{q_L^*}\tilde{\psi}_2(\lambda q)^{1/\gamma_2}\phi_L\left(q\right)dq. \tag{29}$$

A worker of type q_L employed in sector i earns the salary

$$w(q_L) = w_i \tilde{\psi}_i (\lambda q_L)^{1/\gamma_i} \ ,$$

so wage continuity at the marginal worker q_L^* requires

$$w_1 \tilde{\psi}_1 (\lambda q_L^*)^{1/\gamma_1} = w_2 \tilde{\psi}_2 (\lambda q_L^*)^{1/\gamma_2} . \tag{30}$$

Finally, profits for a firm in industry i that hires workers with index q_L are

$$\tilde{\pi}_i (q_L) = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \left[\tilde{\psi}_i (\lambda q_L) \right]^{\frac{1}{1-\gamma_i}} w(q_L)^{-\frac{\gamma_i}{1-\gamma_i}} - r$$

and free entry in both sectors implies

$$\bar{\gamma}_1 p_1^{\frac{1}{1-\gamma_1}} w_1^{-\frac{\gamma_1}{1-\gamma_1}} = \bar{\gamma}_2 p_2^{\frac{1}{1-\gamma_2}} w_2^{-\frac{\gamma_2}{1-\gamma_2}} . \tag{31}$$

Equations (28) - (31) determine q_L^*, H_1, w_1 and w_2.

Now we define $Q_L^* = \lambda q_L^*$ and totally differentiate the equilibrium system (evaluated at $\lambda = 1$), which yields

$$\begin{pmatrix} 1 & -1 & s_L(q_L^*) & 0 \\ -\frac{\gamma_1}{1-\gamma_1} & \frac{\gamma_2}{1-\gamma_2} & 0 & 0 \\ 0 & \frac{E_2}{1-\gamma_2} & \bar{L} \tilde{\psi}_2 (q_L^*)^{1/\gamma_2} \phi_L(q_L^*) q_L^* & E_2 H_1 / H_2 \\ \frac{E_1}{1-\gamma_1} & 0 & -\bar{L} \tilde{\psi}_1 (\lambda q_L^*)^{1/\gamma_1} \phi_L(q_L^*) q_L^* & -E_1 \end{pmatrix} \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{Q}_L^* \\ \hat{H}_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ -\frac{E_2}{\gamma_2} \bar{\varepsilon}_{\tilde{\psi}_2}(q_L^*) + \bar{L} \tilde{\psi}_2 (q_L^*)^{1/\gamma_2} \phi_L(q_L^*) q_L^* \\ -\frac{E_1}{\gamma_1} \bar{\varepsilon}_{\tilde{\psi}_1}(q_L^*) - \bar{L} \tilde{\psi}_1 (q_L^*)^{1/\gamma_1} \phi_L(q_L^*) q_L^* \end{pmatrix} \hat{\lambda},$$

where $\bar{\varepsilon}_{\tilde{\psi}_i}(q_L^*)$ is a weighted average of the elasticities $\varepsilon_{\tilde{\psi}_i}(q_L)$ in sector i, with weights

$$v_i(q_L) = \frac{\tilde{\psi}_i (q_L)^{1/\gamma_i} \phi_L(q_L)}{\int_{q_L \in Q_{Li}} \tilde{\psi}_i (q_L)^{1/\gamma_i} \phi_L(q_L) \, dq_L}, \quad i = 1, 2.$$

Therefore,

$$\begin{aligned} \hat{H}_1 D_{ho} (1-\gamma_1)(1-\gamma_2) = {} & E_1 E_2 s_L(q_L^*) \left[\bar{\varepsilon}_{\tilde{\psi}_1}(q_L^*) - \bar{\varepsilon}_{\tilde{\psi}_2}(q_L^*) \right] \hat{\lambda} \\ & + \bar{L} \phi_L(q_L^*) q_L^* \left[\tilde{\psi}_1 (q_L^*)^{1/\gamma_1} E_2 + \tilde{\psi}_2 (q_L^*)^{1/\gamma_2} E_1 \right] \left[\bar{\varepsilon}_{\tilde{\psi}_1}(q_L^*) - \bar{\varepsilon}_{\tilde{\psi}_2}(q_L^*) \right] \hat{\lambda}. \end{aligned}$$

It follows that, given $s_L(q_L^*) > 0$, an increase in λ raises H_1 if and only if $\bar{\varepsilon}_{\tilde{\psi}_1}(q_L^*) > \bar{\varepsilon}_{\tilde{\psi}_2}(q_L^*)$. Moreover, $\bar{\varepsilon}_{\tilde{\psi}_1}(q_L^*) > \bar{\varepsilon}_{\tilde{\psi}_2}(q_L^*)$ if $\varepsilon_{\tilde{\psi}_1}(q_L') > \varepsilon_{\tilde{\psi}_2}(q_L'')$ for all $q_L', q_L'' \in S_L^A \cup S_L^B$ and $\bar{\varepsilon}_{\tilde{\psi}_1}(q_L^*) < \bar{\varepsilon}_{\tilde{\psi}_2}(q_L^*)$ if $\varepsilon_{\tilde{\psi}_1}(q_L') <$

$\varepsilon_{\tilde{\psi}_2}(q''_L)$ for all $q'_L, q''_L \in S_L^A \cup S_L^B$. This proves Proposition 3.

Proofs for Section 4

Denote by $m_i(q_H)$ the solution set to problem (15). Because S_L and S_H are compact, $m_i(q_H)$ is upper hemicontinuous (because $\tilde{\pi}_i(q_L, q_H)$ is a continuous function), and $m_i(q_H)$ is closed-valued, the graph

$$G_i = [\{q_H, q_L\} \mid q_L \in m_i(q_H) \text{ for all } q_H \in S_H]$$

is closed. The matching correspondence satisfies

$$m(q_H) = \begin{cases} m_1(q_H) & \text{for } q_H \in Q_{H1} \\ m_2(q_H) & \text{for } q_H \in Q_{H2} \end{cases}$$

and the equilibrium allocation graph in sector i is

$$M_i = [\{q_H, q_L\} \mid q_L \in m_i(q_H) \text{ for all } q_H \in Q_{Hi}] \subseteq G_i.$$

Since $Q_{Hi} \subseteq S_H$, the graph M_i is also closed.

Now consider a connected subset $M_i^n \subseteq M_i$:

$$M_i^n = [\{q_H, q_L\} \mid q_L \in m_i(q_H) \text{ for all } q_H \in [q_{H1}, q_{H2}] \subseteq Q_{Hi}].$$

Since M_i is a closed graph, such a subset exists and there exists an interval $[q_{L1}, q_{L2}]$, $q_{L2} > q_{L1}$, that satisfies both (i) $m_i(q_H) \in [q_{L1}, q_{L2}]$ for all $q_H \in [q_{H1}, q_{H2}]$ and (ii) for every point $q_L \in [q_{L1}, q_{L2}]$ there exists a managerial ability level $q_H \in [q_{H1}, q_{H2}]$ satisfying $q_L \in m_i(q_H)$. This means that, in M_i^n, workers of ability $[q_{L1}, q_{L2}]$ are matched with managers of ability $[q_{H1}, q_{H2}]$ and all workers and managers have matches. Then, as Eeckhout and Kircher (2012) have shown, strict log supermodularity of $\psi_i(\cdot)$ ensures strict positive assortative matching (PAM) between the factors allocated to sector i. It follows that $m_i(q_H)$ is a continuous and strictly increasing function in the interior of $[q_{H1}, q_{H2}]$. M_i consists of a union of connected sets, $M_i = \cup_{n \in N_i} M_i^n$, such that $m_i(q_H)$ is continuous and strictly increasing in each such set and $m_i(q_H)$ jumps upwards between them.

We now establish the differentiability of $w(\cdot)$ in $M_i^{n,int}$.[40] Let $m^{-1}(\cdot)$ be the inverse of the sectoral matching function in $M_i^{n,int}$. Since $m(\cdot)$ is continuous and strictly increasing in $M_i^{n,int}$, this inverse exists. Now consider an interval $[q'_L, q'_L + dq_L) \in M_i^{n,int}$. The zero-profit condition (16) implies

$$w(q'_L) = \bar{\gamma}_i^{\frac{1-\gamma_i}{\gamma_i}} p_i^{\frac{1}{\gamma_i}} \psi_i \left[m^{-1}(q'_L), q'_L\right]^{\frac{1}{\gamma_i}} r \left[m^{-1}(q'_L)\right]^{-\frac{1-\gamma_i}{\gamma_i}}$$

and profit maximization implies

$$w(q'_L + dq_L) \geq \bar{\gamma}_i^{\frac{1-\gamma_i}{\gamma_i}} p_i^{\frac{1}{\gamma_i}} \psi_i \left[m^{-1}(q'_L), q'_L + dq_L\right]^{\frac{1}{\gamma_i}} r \left[m^{-1}(q'_L)\right]^{-\frac{1-\gamma_i}{\gamma_i}}.$$

[40] This proof is similar to the proof of differentiability of the wage function in Sampson (2012).

Together, these expressions imply

$$w\left(q_L'+dq_L\right)\geq w\left(q_L'\right)\left\{\frac{\psi_i\left[m^{-1}\left(q_L'\right),q_L'+dq_L\right]}{\psi_i\left[m^{-1}\left(q_L'\right),q_L'\right]}\right\}^{\frac{1}{\gamma_i}}. \tag{32}$$

Similarly, (16) implies

$$w\left(q_L'+dq_L\right)=\bar{\gamma}_i^{\frac{1-\gamma_i}{\gamma_i}}p_i^{\frac{1}{\gamma_i}}\psi_i\left[m^{-1}\left(q_L'+dq_L\right),q_L'+dq_L\right]^{\frac{1}{\gamma_i}}r\left[m^{-1}\left(q_L'+dq_L\right)\right]^{-\frac{1-\gamma_i}{\gamma_i}}$$

and profit maximization implies

$$w\left(q_L'\right)\geq\bar{\gamma}_i^{\frac{1-\gamma_i}{\gamma_i}}p_i^{\frac{1}{\gamma_i}}\psi_i\left[m^{-1}\left(q_L'+dq_L\right),q_L'\right]^{\frac{1}{\gamma_i}}r\left[m^{-1}\left(q_L'+dq_L\right)\right]^{-\frac{1-\gamma_i}{\gamma_i}}.$$

Together, these expressions imply

$$w\left(q_L'\right)\geq w\left(q_L'+dq_L\right)\left\{\frac{\psi_i\left[m^{-1}\left(q_L'+dq_L\right),q_L'\right]}{\psi_i\left[m^{-1}\left(q_L'+dq_L\right),q_L'+dq_L\right]}\right\}^{\frac{1}{\gamma_i}}. \tag{33}$$

Inequalities (32) and (33) jointly imply

$$\frac{w\left(q_L'\right)}{\psi_i\left[m^{-1}\left(q_L'\right),q_L'\right]^{\frac{1}{\gamma_i}}}\left[\frac{\psi_i\left[m^{-1}\left(q_L'\right),q_L'+dq_L\right]^{\frac{1}{\gamma_i}}-\psi_i\left[m^{-1}\left(q_L'\right),q_L'\right]^{\frac{1}{\gamma_i}}}{dq_L}\right]\leq\frac{w\left(q_L'+dq_L\right)-w\left(q_L'\right)}{dq_L}$$
$$\leq\frac{w\left(q_L'\right)}{\psi_i\left[m^{-1}\left(q_L'+dq_L\right),q_L'\right]^{\frac{1}{\gamma_i}}}\left[\frac{\psi_i\left[m^{-1}\left(q_L'+dq_L\right),q_L'+dq_L\right]^{\frac{1}{\gamma_i}}-\psi_i\left[m^{-1}\left(q_L'+dq_L\right),q_L'\right]^{\frac{1}{\gamma_i}}}{dq_L}\right].$$

Since, by Assumption 3, the productivity function is continuous, strictly increasing, and differentiable, and since the inverse of the sectoral matching function is continuous and strictly increasing in this range, taking the limit as $dq_L\to 0$ implies that the derivative of $w(\cdot)$ at q_L' exists and

$$\frac{dw\left(q_L'\right)}{dq_L}=\frac{w\left(q_L'\right)}{\psi_i\left[m^{-1}\left(q_L'\right),q_L'\right]^{\frac{1}{\gamma_i}}}\frac{\partial\psi_i\left[m^{-1}\left(q_L'\right),q_L'\right]^{\frac{1}{\gamma_i}}}{\partial q_L}.$$

Similar arguments can be used to show that the salary function is differentiable.

We now prove Proposition 6 by contradiction. (Proposition 5 can be proved similarly.) To this end, suppose that the inequality condition holds, but the equilibrium is such that there are managers employed in sector j who have greater ability than some managers employed in sector i. In such circumstances, there exists an ability level $\tilde{q}_H$ at one of the boundaries between Q_{Hi} and Q_{Hj} such that managers with ability in $\left(\tilde{q}_H-\varepsilon_i,\tilde{q}_H\right)\subset Q_{Hi}^{int}$ are employed in sector i and managers with ability $\left(\tilde{q}_H,\tilde{q}_H+\varepsilon_j\right)\subset Q_{Hj}^{int}$ are employed in sector j, for $\varepsilon_i>0$ and $\varepsilon_j>0$ small enough. Moreover, the equilibrium conditions (16)-(18) are satisfied, the matching function $m\left(q_H\right)$ is continuous at Q_{Hi}^{int} and Q_{Hj}^{int} close to $\tilde{q}_H$ (but can be discontinuous at the boundary point between these sets), the wage function $w\left(q_L\right)$ is continuous and increasing in S_L and differentiable in Q_{Li}^{int} and Q_{Lj}^{int}, and the salary function $r\left(q_H\right)$ is continuous and increasing in S_H and differentiable in Q_{Hi}^{int} and Q_{Hj}^{int}.

Now recall the continuous profit function $\Pi_i(q_H)$ defined in (15). In equilibrium, $\Pi_i(q_H) = 0$ for all $q_H \in Q_{Hi}$, but the maximal profits $\Pi_i(q_H)$ may differ from zero for $q_H \notin Q_{Hi}$. Therefore $\Pi_i(q_H) = 0$ for all $q_H \in (\tilde{q}_H - \varepsilon_i, \tilde{q}_H)$ and, by continuity, $\lim_{q_H \nearrow \tilde{q}_H} \Pi_i(q_H) = 0$.

Next consider the profits that would accrue to an entrepreneur that hires a manager with ability $\tilde{q}_H + \varepsilon$ in order to produce good i, where $\varepsilon < \varepsilon_j$. Choosing workers so as to maximize profits, this entrepreneur earns $\Pi_i(\tilde{q}_H + \varepsilon) \geq \pi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]$, where $m\left(\tilde{q}_H^-\right) = \lim_{\varepsilon \searrow 0} m(\tilde{q}_H - \varepsilon)$ and $\lim_{\varepsilon \searrow 0} \Pi_i(\tilde{q}_H + \varepsilon) = \lim_{\varepsilon \searrow 0} \pi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right] = 0$. The first-order approximation to $\pi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]$ is

$$\pi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right] \approx \varepsilon \pi_{iH}\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right],$$

where $\pi_{iH}(\cdot)$ is the partial derivative of $\pi_i(\cdot)$ with respect to q_H. This derivative exists because the salary function is differentiable in Q_{Hj}^{int}, and

$$\begin{aligned}
&\pi_{iH}\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right] \\
&= \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \psi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]^{\frac{1}{1-\gamma_i}} w\left[m\left(\tilde{q}_H^-\right)\right]^{-\frac{\gamma_i}{1-\gamma_i}} \frac{\psi_{iH}\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]}{(1-\gamma_i)\psi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]} - r'(\tilde{q}_H + \varepsilon) \\
&= \left\{\frac{\psi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]}{\psi_i\left[\tilde{q}_H, m\left(\tilde{q}_H^-\right)\right]}\right\}^{\frac{1}{1-\gamma_i}} r\left(\tilde{q}_H^-\right) \frac{\psi_{iH}\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]}{(1-\gamma_i)\psi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]} - r'(\tilde{q}_H + \varepsilon),
\end{aligned}$$

where the last equality uses the free-entry condition (16), which applies to sector 1 at points in Q_{Hi}^{int} in the conjectured equilibrium, and $r\left(\tilde{q}_H^-\right) = r(\tilde{q}_H)$ due to the continuity of the salary function. Since $\tilde{q}_H + \varepsilon \in Q_{Hj}^{int}$, condition (18) implies

$$\lim_{\varepsilon \searrow 0} \pi_{iH}\left[\tilde{q}_H + \varepsilon, m_i\left(\tilde{q}_H^-\right)\right] = r(\tilde{q}_H)\left\{\frac{\psi_{iH}\left[q_H, m_i\left(\tilde{q}_H^-\right)\right]}{(1-\gamma_i)\psi_i\left[q_H, m_i\left(\tilde{q}_H^-\right)\right]} - \frac{\psi_{jH}\left[q_H, m\left(\tilde{q}_H^+\right)\right]}{(1-\gamma_j)\psi_j\left[q_H, m\left(\tilde{q}_H^+\right)\right]}\right\},$$

where $m\left(\tilde{q}_H^+\right) = \lim_{\varepsilon \searrow 0} m(\tilde{q}_H + \varepsilon)$. It now follows from supposition of Proposition 6 that the right-hand side of this equation is strictly positive irrespective of the values of $m_i\left(\tilde{q}_H^-\right)$ and $m\left(\tilde{q}_H^+\right)$, and therefore that $\pi_{iH}\left[\tilde{q}_H + \varepsilon, m_i\left(\tilde{q}_H^-\right)\right] > 0$ for ε small enough, which contradicts the zero-profit condition as profits rise above zero This contradicts the supposition that in equilibrium there are managers employed in sector j who are more able than some managers employed in sector i. Consequently, every manager in sector i has greater ability than any manager employed in sector j. This completes the proof.

Next we prove Proposition 7. Suppose that the inequality conditions in Proposition 7 hold but the equilibrium is such that there exist managers in sector 2 who are more able than some managers in sector 1. In such circumstances, there exists an ability $\tilde{q}_H$ at one of the boundary points between Q_{H1} and Q_{H2} such that managers of ability $\tilde{q}_H - \varepsilon_1$ are employed in sector 1 and managers of ability $\tilde{q}_H + \varepsilon_2$ are employed in sector 2 for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ small enough. Let $m(\tilde{q}_H^-) = \lim_{q_H \nearrow \tilde{q}_H} m(q_H)$ and $m(\tilde{q}_H^+) = \lim_{q_H \searrow \tilde{q}_H} m(q_H)$ Then

$$\lim_{\varepsilon \to 0} \pi_{iH}\left[\tilde{q}_H + \varepsilon, m(q_H^-)\right] = r(\tilde{q}_H)\left[\frac{\psi_{1H}\left[\tilde{q}_H, m(\tilde{q}_H^-)\right]}{(1-\gamma_1)\psi_1\left[\tilde{q}_H, m(\tilde{q}_H^-)\right]} - \frac{\psi_{2H}\left[\tilde{q}_H, m\left(\tilde{q}_H^+\right)\right]}{(1-\gamma_2)\psi_2\left[\tilde{q}_H, m\left(\tilde{q}_H^+\right)\right]}\right], \tag{34}$$

which we derive in the same way as in the proof of Proposition 6. Under the supposition that the managers to the left of $\tilde{q}_H$ sort into sector 1 and those to the right of $\tilde{q}_H$ sort into sector 2 the partial derivative in

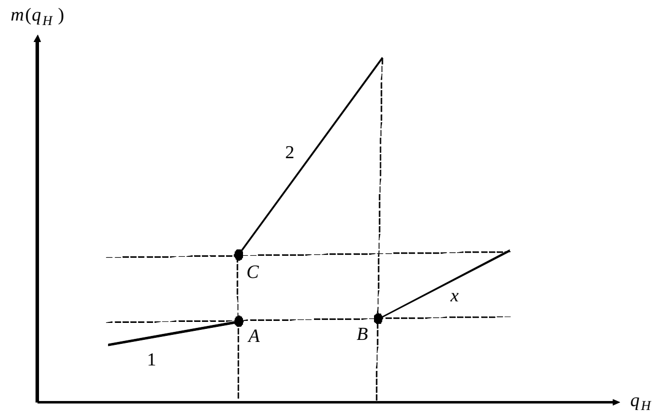

Figure 11: Matching function with discontinuity

(34) cannot be positive and therefore

$$\frac{\psi_{1H}\left[\tilde{q}_H, m(\tilde{q}_H^-)\right]}{(1-\gamma_1)\,\psi_1\left[\tilde{q}_H, m(\tilde{q}_H^-)\right]} \leq \frac{\psi_{2H}\left[\tilde{q}_H, m\left(\tilde{q}_H^+\right)\right]}{(1-\gamma_2)\,\psi_2\left[\tilde{q}_H, m\left(\tilde{q}_H^+\right)\right]}.$$

In view of the first inequality in Proposition 7 and the strict log supermodularity of the productivity function, this inequality implies $m\left(\tilde{q}_H^+\right) > m\left(\tilde{q}_H^-\right)$. That is, the matching function is discontinuous at $\tilde{q}_H$ and it jumps upwards there. As a result, there must exist an ability level for workers $\check{q}_L \in \left[m\left(\tilde{q}_H^-\right), m\left(\tilde{q}_H^+\right)\right]$ such that workers in the range $(\check{q}_L - \breve{\varepsilon}_1, \check{q}_L)$ are employed in sector 1 and workers in the range $(\check{q}_L, \check{q}_L + \breve{\varepsilon}_2)$ are employed in sector 2, for $\breve{\varepsilon}_1$ and $\breve{\varepsilon}_2$ small enough. Due to the upward jump of the matching function and due to PAM in each sector, in this range of worker types the ability of managers matched with workers in sector 1 must be strictly greater than the ability of managers matched with workers in sector 2. This is illustrated in Figure 11. At point A, we have $q_H = \tilde{q}_H$ and the matching function exhibits an upward jump from point A to C. The supposition is that managers to the left of A sort into sector 1 and managers to the right of A sort into sector 2, as illustrated in the figure. Clearly, workers with ability between points A and C must be matched with managers in some sector. Segment x illustrates a possible matching of these workers with high-ability managers. It is not possible for x to be sector 2, however, because this would imply non-monotonic matching in this sector, which is ruled out by the strict log supermodularity of the productivity function there. So x must be sector 1. In this case, $\check{q}_L$ is the ability of workers at point C. Workers with ability just below C are employed in sector 1 and workers with ability just above C are employed in sector 2. Evidently, the ability of managers with whom these workers are matched in sector 1 is higher than the ability of managers with whom their slightly better peers are matched in sector 2. It can be seen from the figure that a similar outcome obtains if the matching along x is to the left of point A, except that in this case x stands for sector 2 and $\check{q}_L$ is the ability of workers at point A. Evidently, in this case too, at points around $\check{q}_L$ the ability of managers matched with workers in sector 1 is higher than the ability of managers matched with workers in sector 2.

In short, consider the inverse function $m_1^{-1}(q_L)$ for $q_L \in (\check{q}_L - \breve{\varepsilon}_1, \check{q}_L)$; this inverse exists in the specified range because $m_1(q_H)$ is continuous and strictly increasing at points in $(\tilde{q}_H - \varepsilon, \tilde{q}_H)$ for ε small enough. Similarly, consider the inverse function $m_2^{-1}(q_L)$ for $q_L \in (\check{q}_L, \check{q}_L + \check{\varepsilon}_2)$; this inverse also exists in the specified range because $m_2(q_H)$ is continuous and strictly increasing at points in $(\tilde{q}_H, \tilde{q}_H + \varepsilon)$ for ε small enough. Moreover, under the supposition of our sorting pattern $m^{-1}(q_L) = m_1^{-1}(q_L)$ for $q_L \in (\check{q}_L - \check{\varepsilon}_1, \check{q}_L)$ and $m^{-1}(q_L) = m_2^{-1}(q_L)$ for $q_L \in (\check{q}_L, \check{q}_L + \check{\varepsilon}_2)$ and the argument in the previous paragraph showed that $m^{-1}(q_L) = m_1^{-1}(q_L) > m^{-1}(q_L') = m_2^{-1}(q_L')$ for $q_L \in (\check{q}_L - \check{\varepsilon}_1, \check{q}_L)$ and $q_L' \in (\check{q}_L, \check{q}_L + \check{\varepsilon}_2)$. Taking limits as $\check{\varepsilon}_1, \check{\varepsilon}_2 \searrow 0$, this implies that $m^{-1}\left(\check{q}_L^-\right) > m^{-1}\left(\check{q}_L^+\right)$.

Next, following steps similar to those used in the proof of Proposition 6, which considered the response of profits to variations in the ability of managers at points around $\tilde{q}_H$, an analysis of the response of profits to variations in the ability of workers at points around $\check{q}_L$ establishes that a necessary condition for optimality is

$$\frac{\psi_{1L}\left[m^{-1}\left(\check{q}_L^-\right), \check{q}_L\right]}{\gamma_1\psi_1\left[m^{-1}\left(\check{q}_L^-\right), \check{q}_L\right]} \leq \frac{\psi_{2L}\left[m^{-1}\left(\check{q}_L^+\right), \check{q}_L\right]}{\gamma_2\psi_2\left[m^{-1}\left(\check{q}_L^+\right), \check{q}_L\right]}.$$

In view of the second inequality in Proposition 7 and the strict log supermodularity of the productivity function, this inequality implies $m^{-1}\left(\check{q}_L^+\right) = m_2^{-1}\left(\check{q}_L^+\right) > m_1^{-1}\left(\check{q}_L^-\right) = m^{-1}\left(\check{q}_L^-\right)$, which contradicts the above established result that $m_1^{-1}\left(\check{q}_L^-\right) > m_2^{-1}\left(\check{q}_L^+\right)$. It follows that the best managers sort into sector 1. By symmetrical arguments the best workers also sort into sector 1.

Matching and Factor Prices Among a Group of Workers and Managers

In order to prove the remaining propositions in the main text, we need to understand how matching within an allocation set and the wages and salaries of workers and managers in the set respond to changes in factor endowments, the price of the output produced by these factors, and the boundaries of workers' and managers' abilities.

Suppose that some sector employs workers and managers whose abilities form the intervals $S_L = [q_{La}, q_{Lb}]$ and $S_H = [q_{Ha}, q_{Hb}]$. To simplify notation, we drop the sectoral index i and denote q_H by q, and we consider the following industry equilibrium conditions:

$$r(q) = \bar{\gamma} p^{\frac{1}{1-\gamma}} \psi[q, m(q)]^{\frac{1}{1-\gamma}} w[m(q)]^{-\frac{\gamma}{1-\gamma}}, \quad \bar{\gamma} = \gamma^{\frac{\gamma}{1-\gamma}}(1-\gamma) \tag{35}$$

$$\frac{\psi_L[q, m(q)]}{\gamma\psi[q, m(q)]} = \frac{w'[m(q)]}{w[m(q)]}, \tag{36}$$

$$\bar{H}\frac{\gamma r(q)}{(1-\gamma) w[m(q)]}\phi_H(q) = \bar{L}\phi_L[m(q)]\, m'(q), \tag{37}$$

and the boundary conditions,

$$\begin{aligned} m(q_{Hz}) &= q_{Lz}, \quad z = a, b; \\ q_{Lb} &> q_{La} > 0, \quad q_{Hb} > q_{Ha} > 0. \end{aligned} \tag{38}$$

Equation (35) is taken from (16), (36) is taken from (17) and (37) is taken from (19). We seek to characterize the solution for the three functions, $w(\cdot)$, $r(\cdot)$ and $m(\cdot)$.

We use (35) and (36) to obtain

$$\ln r(q_H) - \ln r(q_{H0}) = \int_{q_{H0}}^{q_H} \frac{\psi_H[x, m(x)]}{(1-\gamma)\psi[x, m(x)]} dx, \quad \text{for } q_H, q_{H0} \in S_H, \tag{39}$$

$$\ln w(q_L) - \ln w(q_{L0}) = \int_{q_{L0}}^{q_L} \frac{\psi_L[\mu(x), x]}{\gamma\psi[\mu(x), x]} dx, \quad \text{for } q_L, q_{L0} \in S_L, \tag{40}$$

where $\mu(\cdot)$ is the inverse of $m(\cdot)$. We substitute (35) into (37) to obtain

$$\begin{aligned} \frac{1}{1-\gamma} \ln w[m(q)] \quad &= \quad \frac{1}{1-\gamma} \ln \gamma + \ln\left(\frac{\bar{H}}{\bar{L}}\right) + \frac{1}{1-\gamma} \ln p \\ &\quad + \frac{1}{1-\gamma} \ln \psi[q, m(q)] + \log \phi_H(q) - \log \phi_L[m(q)] - \log m'(q). \end{aligned} \tag{41}$$

The differential equations (36) and (41) together with the boundary conditions (38) uniquely determine the solution of $w(\cdot)$ and $m(\cdot)$ when the productivity function $\psi(\cdot)$ is twice continuously differentiable and the density functions $\phi_F(\cdot)$, $F = H, L$, are continuously differentiable.

By differentiating (41) and substituting (36) into the result, we generate a second-order differential equation for the matching function,

$$\frac{m''(q)}{m'(q)} = \frac{\psi_H[q, m(q)]}{(1-\gamma)\psi_L[q, m(q)]} - \frac{\psi_L[q, m(q)]\, m'(q)}{\gamma\psi[q, m(q)]} + \frac{\phi'_H(q)}{\phi_H(q)} - \frac{\phi'_L[m(q)]\, m'(q)}{\phi_L[m(q)]}. \tag{42}$$

Given boundary conditions $m(q_0) = q_{L0}$, $m'(q_0) = t_0 > 0$, this differential equation has a unique solution, which may or may not satisfy the boundary conditions (38). The solution to the original matching problem is found by identifying a value t_a such that $m(q_{Ha}) = q_{La}$ and $m'(q_{Ha}) = t_a$ yield a solution that satisfies the second boundary condition $m(q_{Hb}) = q_{Lb}$. Note that this solution depends neither on the price p nor on the factor endowments $\bar{H}$ and $\bar{L}$. Therefore, changes in these variables do no affect the matching function, but they change all wages and salaries proportionately, as can be seen from (41), and (35). We have

Lemma 1 *(i) The matching function $m(\cdot)$ does not depend on $(p, \bar{H}, \bar{L})$. (ii) An increase in the price p, $\hat{p} > 0$, raises the wage and salary schedules proportionately by $\hat{p}$. (iii) An increase in $\bar{H}/\bar{L}$ such that $\hat{H} - \hat{L} = \hat{\eta} > 0$ raises the wage schedule proportionately by $(1-\gamma)\hat{\eta}$ and reduces the salary schedule proportionately by $\gamma\hat{\eta}$.*

We now prove several lemmas that are used in the main analysis.

Lemma 2 *Let $[m_\varkappa(q), w_\varkappa(q_L)]$ and $[m_\varrho(q), w_\varrho(q_L)]$ be solutions to the differential equations (36) and (41), each for different boundary conditions (38), such that $m_\varkappa(q_0) = m_\varrho(q_0) = q_{L0}$ and $m'_\varrho(q_0) > m'_\varkappa(q_0)$ for $q_0 \in S_{H\varkappa} \cap S_{H\varrho}$. Then $m_\varrho(q) > m_\varkappa(q)$ for all $q > q_0$ and $m_\varrho(q) < m_\varkappa(q)$ for all $q < q_0$ in the overlapping range of abilities.*

Proof. Consider $q > q_0$ and suppose that, contrary to the claim, there exists a $q_1 > q_0$ such that $m_\varrho(q_1) \leq m_\varkappa(q_1)$. Then differentiability of $m_\iota(\cdot)$, $\iota = \varkappa, \varrho$, implies that there exists $q_2 > q_0$ such that $m_\varrho(q_2) = m_\varkappa(q_2)$, $m_\varrho(q) > m_\varkappa(q)$ for all $q \in (q_0, q_2)$ and $m'_\varrho(q_2) < m'_\varkappa(q_2)$. This also implies $\mu_\varrho(x) < \mu_\varkappa(x)$

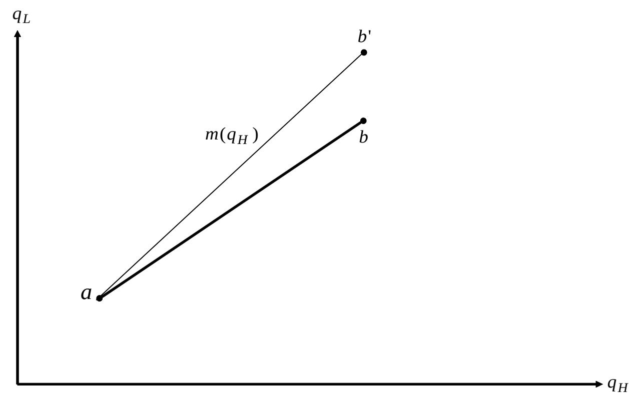

Figure 12: Shift in the matching function when q_L^b rises to $q_L^{b'}$

for all $x \in (m_\varrho(q_0), m_\varrho(q_2))$, where $\mu_\iota(\cdot)$ is the inverse of $m_\iota(\cdot)$. Under these conditions (41) implies $w_\varrho[m_\varrho(q_0)] < w_\varkappa[m_\varrho(q_0)]$ and $w_\varrho[m_\varrho(q_2)] > w_\varkappa[m_\varrho(q_2)]$, and therefore

$$w_\varkappa[m_\varrho(q_2)] - w_\varkappa[m_\varrho(q_0)] < w_\varrho[m_\varrho(q_2)] - w_\varrho[m_\varrho(q_0)].$$

On the other hand, (40) implies

$$\ln w_\iota[m_\varrho(q_2)] - \ln w_\iota[m_\varrho(q_0)] = \int_{m_\varrho(q_0)}^{m_\varrho(q_2)} \frac{\psi_L[\mu_\iota(x), x]}{\gamma\psi[\mu_\iota(x), x]} dx, \quad \iota = \varkappa, \varrho.$$

Together with the previous inequality, this gives

$$\int_{m_\varrho(q_0)}^{m_\varrho(q_2)} \frac{\psi_L[\mu_\varkappa(x), x]}{\psi[\mu_\varkappa(x), x]} dx < \int_{m_\varrho(q_0)}^{m_\varrho(q_2)} \frac{\psi_L[\mu_\varrho(x), x]}{\psi[\mu_\varrho(x), x]} dx.$$

Note, however, that strict log supermodularity of $\psi(\cdot)$ and $\mu_\varrho(x) < \mu_\varkappa(x)$ for all $x \in (m_\varrho(q_0), m_\varrho(q_2))$ imply the reverse inequality, a contradiction. It follows that $m_\varrho(q) > m_\varkappa(q)$ for all $q > q_0$. A similar argument shows that $m_\varrho(q) < m_\varkappa(q)$ for all $q < q_0$. ■

The key implication of this lemma is that changes in the boundary conditions (38) shift the matching function in such a way as to generate at most one point in common with the original matching function. We next show how the matching function and wage function respond to the boundary conditions. To this end, re-consider Figure 4 in the main text. Let the thick curve between points a and b represent the solution to the matching function when points a and b are the boundary points (38). Now consider the shift of the equilibrium matching function in response to a rise in q_{Lb}; that is, the end point b shifts upward to b'. Since point a is common to the old and new matching function, Lemma 2 implies that the two curves can have no additional points in common, which implies that the new inverse matching function—represented by the thin curve between points a and b'—is everywhere above the old one. It follows that an increase in q_{Lb} increases the ability of workers matched with every manager except for the least able manager. Other shifts in the boundary points can be analyzed in similar fashion to establish

Lemma 3 *(i) $dm(q_H)/dq_{La} > 0$ for all $q_H < q_{Hb}$ and $d\mu(q_L)/dq_{La} < 0$ for all $q_L < q_{Lb}$; (ii) $dm(q_H)/dq_{Lb} > 0$ for all $q_H > q_{Ha}$ and $d\mu(q_L)/dq_{Lb} < 0$ for all $q_L > q_{La}$; (iii) $d\mu(q_L)/dq_{Ha} > 0$ for all $q_L < q_{Lb}$ and $dm(q_H)/dq_{Ha} < 0$ for all $q_H < q_{Hb}$; and (iv) $d\mu(q_L)/dq_{Hb} > 0$ for all $q_L > q_{La}$ and $dm(q_H)/dq_{Hb} < 0$ for all $q_H > q_{Ha}$.*

The rule that emerges from this lemma is that an improvement in the ability of workers at a boundary of S_L improves the quality of the matches for all the managers (except those at the other boundary) and deteriorates the quality of the matches for all the workers (except those at the other boundary). Similarly, an improvement in the ability of managers at a boundary of S_H improves the quality of the matches for all workers (except those at the other boundary) and deteriorates the quality of the matches for all the managers (except those at the other boundary).

Next consider changes in a boundary (q_{Hz}, q_{Lz}), $z = a, b$. For concreteness, suppose that (q_{Hb}, q_{Lb}) changes. Then the new matching function coincides with the old one at the other boundary point, (q_{Ha}, q_{La}), which has not changed. In this case, Lemma 2 implies that either the two matching functions coincide in the overlapping range of abilities or one is above the other everywhere except for at (q_{Ha}, q_{La}). A similar argument applies to changes in (q_{Ha}, q_{La}). We therefore have:

Lemma 4 *In response to a shift in a single boundary (q_{Hz}, q_{Lz}), $z = a, b$, either the new matching functions coincide with the old matching function in the overlapping range of abilities or one matching function is above the other everywhere except for at the opposite boundary point.*

We next discuss the impact of boundaries on wages and salaries. We focus on wages, but note that if a shift in boundaries raises the wage of workers with ability q_L then it must reduce the salary of managers teamed with these workers. This can be seen from (35) by noting that a change in boundaries has no impact on $r(\cdot)$ through an induced shift in the matching function due to the first-order condition (36) (a version of the Envelope Theorem). Therefore the change in salary $r(q)$ is driven by the change in wages of workers matched with managers of ability q. We record this result in

Lemma 5 *Suppose that the boundaries (q_{Hz}, q_{Lz}), $z = a, b$, change and that, as a result, $w(q_L)$ rises for some q_L such that q_L and $q = m^{-1}(q_L)$ are in the overlapping range of abilities of the old and new boundaries. Then $r(q)$ declines.*

For the subsequent analysis the following lemma is useful:

Lemma 6 *Let $[m_\varkappa(q), w_\varkappa(q_L)]$ and $[m_\varrho(q), w_\varrho(q_L)]$ be solutions to the differential equations (36) and (41), each for different boundary conditions (38), such that $m_\varkappa(q_0) = m_\varrho(q_0) = q_{L0}$ and $m'_\varrho(q_0) > m'_\varkappa(q_0)$ for some $q_0 \in S_{L\varkappa} \cap S_{L\varrho}$, and let $r_\varrho(q)$ and $r_\varkappa(q)$ be the corresponding solutions to (35). Then $w_\varrho(q_L) < w_\varkappa(q_L)$ and $r_\varrho(q) > r_\varkappa(q)$ in the overlapping range of abilities.*

Proof. From Lemma 2 we know that $m_{\varrho}(q) > m_{\varkappa}(q)$ for all $q > q_0$ and $m_{\varrho}(q) < m_{\varkappa}(q)$ for all $q < q_0$ in the overlapping range of abilities and $\mu_{\varrho}(x) < \mu_{\varkappa}(x)$ for all $x > q_{L0}$ and $\mu_{\varrho}(x) > \mu_{\varkappa}(x)$ for all $x < q_{L0}$ in the overlapping range of abilities. Moreover, $m'_{\varrho}(q_0) > m'_{\varkappa}(q_0)$ and (41) imply

$$\ln w_{\varkappa}(q_{L0}) > \ln w_{\varrho}(q_{L0})$$

while (40) implies

$$\ln w_{\iota}(q_L) - \ln w_{\iota}(q_{L0}) = \int_{q_{L0}}^{q_L} \frac{\psi_L[\mu_{\iota}(x), x]}{\gamma\psi[\mu_{\iota}(x), x]} dx, \quad \iota = \varkappa, \varrho.$$

Together, these inequalities imply

$$\begin{aligned} \ln w_{\varkappa}(q_L) - \ln w_{\varrho}(q_L) &> \int_{q_{L0}}^{q_L} \frac{\psi_L[\mu_{\varkappa}(x), x]}{\gamma\psi[\mu_{\varkappa}(x), x]} dx - \int_{q_{L0}}^{q_L} \frac{\psi_L[\mu_{\varrho}(x), x]}{\gamma\psi[\mu_{\varrho}(x), x]} dx \\ &= \int_{q_L}^{q_{L0}} \frac{\psi_L[\mu_{\varrho}(x), x]}{\gamma\psi[\mu_{\varrho}(x), x]} dx - \int_{q_L}^{q_{L0}} \frac{\psi_L[\mu_{\varkappa}(x), x]}{\gamma\psi[\mu_{\varkappa}(x), x]} dx. \end{aligned}$$

For $q_L > q_{L0}$ the right-hand side of the first line is positive due to the strict log supermodularity of the productivity function and $\mu_{\varrho}(x) < \mu_{\varkappa}(x)$ for all $x > q_{L0}$, and the second line also is positive for $q_L < q_{L0}$ due to the strict log supermodularity of the productivity function and $\mu_{\varrho}(x) > \mu_{\varkappa}(x)$ for all $x < q_{L0}$. It follows that $w_{\varkappa}(q_L) > w_{\varrho}(q_L)$ for all q_L in the overlapping range of abilities. A similar argument establishes that $r_{\varkappa}(q) < r_{\varrho}(q)$ for all q in the overlapping range of abilities. ∎

This lemma, together with Lemma 4, have straightforward implications for the impact of boundary points on the wage and salary functions.

Corollary 1 *Suppose that the lower boundary (q_{Ha}, q_{La}) changes and the matching function shifts upwards as a result. Then salaries decline and wages rise in the overlapping range of abilities. The converse holds when the matching function shifts downwards.*

Corollary 2 *Suppose that the upper boundary (q_{Hb}, q_{Lb}) changes and the matching function shifts upwards as a result. Then salaries rise and wages decline in the overlapping range of abilities. The converse holds when the matching function shifts downwards.*

Not only do wages and salaries shift in a predictable way in response to a shift in a boundary point, the inequality of wages and of salaries also change in predictable ways. From (40) we see that a change in boundaries that shifts upwards the matching function reduces wage inequality, because for every two ability levels the ratio of the wage of a high-ability worker to the wage of a low-ability worker declines for all types in the overlapping range. For salaries it is the opposite, as one can see from (39). We therefore have

Lemma 7 *Suppose that the matching function shifts upwards in response to a shift in the boundaries (38). Then wage inequality narrows and salary inequality widens. The opposite is true when the matching function shifts downwards.*

General Equilibrium

Consider a two-sector economy in which the most-able workers are employed in one sector and the least-able workers are employed in the other sector, and similarly for managers. In such circumstances, the equilibrium can take one of two forms: either the highest-ability workers and highest-ability managers are employed in the same sector and the lowest-ability workers and lowest-ability managers are employed in the other, which we designated as an HH/LL equilibrium, or the highest-ability workers and lowest-ability managers are employed in one sector and the lowest-ability workers and highest-ability managers are employed in the other, which we designated as an HL/LH equilibrium. Our first result is

Lemma 8 *Suppose that the economy has a threshold equilibrium either of the HH/LL or HL/LH type. Then: (i) if the best workers sort into the labor-intensive sector then an increase in $\bar{H}/\bar{L}$ raises the cutoff q_L^* and if the best workers sort into the manager-intensive sector then an increase in $\bar{H}/\bar{L}$ reduces the cutoff q_L^*; and (ii) if the best managers sort into the labor-intensive sector, then an increase in $\bar{H}/\bar{L}$ raises the cutoff q_H^* and if the best managers sort into the manager-intensive sector then an increase in $\bar{H}/\bar{L}$ reduces the cutoff q_H^*.*

To prove this lemma, label the sectors so that the best workers sort into sector 1. We first prove the result for an HH/LL equilibrium and then for an HL/LH equilibrium.

HH/LL Equilibrium

In an HH/LL equilibrium the cutoffs $\{q_H^*, q_L^*\}$ satisfy:

$$w_1\left(q_L^*\right) = w_2\left(q_L^*\right), \tag{43}$$

$$r_1\left(q_H^*\right) = r_2\left(q_H^*\right), \tag{44}$$

where $[w_i(\cdot), r_i(\cdot), m_i(\cdot)]$ is a solution to the single-sector differential equations (36) and (41) for $i = 1, 2$ with the boundary conditions

$$m_2\left(q_{H\min}\right) = q_{L\min}, \quad m_2\left(q_H^*\right) = q_L^*, \tag{45}$$

$$m_1\left(q_H^*\right) = q_L^*, \quad m_1\left(q_{H\max}\right) = q_{L\max}. \tag{46}$$

Clearly, the solutions for the wage function, the salary function, and the matching functions depend on the parameters of the model, such as prices and factor endowments, as do the equilibrium cutoffs $\{q_H^*, q_L^*\}$. We denote by $dw_i(q_L)/d\vartheta$ the derivative of the wage function in sector i with respect to a parameter ϑ, where this derivative accounts for the endogenous adjustments of all three functions. This derivative contrasts with $w_i'(q_L)$, which is the slope of the wage function for given parameters. We use similar notation to represent derivatives of the salary function.

For now, we are interested in $\eta = \bar{H}/\bar{L}$ and we shall use the following elasticities

$$\varepsilon_{w_i,\eta}^* = \left.\frac{dw_i(q_L)}{d\left(\bar{H}/\bar{L}\right)} \cdot \frac{\bar{H}/\bar{L}}{q_L}\right|_{q_L = q_L^*}, \quad \varepsilon_{r_i,\eta}^* = \left.\frac{dr_i(q_H)}{d\left(\bar{H}/\bar{L}\right)} \cdot \frac{\bar{H}/\bar{L}}{q_H}\right|_{q_H = q_H^*}.$$

Differentiating (43)-(44) with respect to $\eta \equiv \bar{H}/\bar{L}$ yields

$$\left[\frac{w_1'\left(q_L^*\right)}{w_1\left(q_L^*\right)} - \frac{w_2'\left(q_L^*\right)}{w_2\left(q_L^*\right)}\right] dq_L^* = \varepsilon_{w_2,\eta}^* - \varepsilon_{w_1,\eta}^*, \tag{47}$$

$$\left[\frac{r_1'(q_H^*)}{r_1(q_H^*)} - \frac{r_2'(q_H^*)}{r_2(q_H^*)}\right] dq_H^* = \varepsilon_{r_2,\eta}^* - \varepsilon_{r_1,\eta}^*. \tag{48}$$

The assumptions that the equilibrium is of the HH/LL type and that the best workers and managers sort into sector 1 imply that the expressions in the square brackets are positive in both equations; that is, at the boundary $\{q_H^*, q_L^*\}$ between the two sectors the slopes of the wage and salary functions have to be steeper in sector 1 into which the more able employees sort. It follows that q_L^* rises in response to an increase in the ratio of managers to workers if and only if $\varepsilon_{w_2,\eta}^* > \varepsilon_{w_1,\eta}^*$ and the cutoff q_H^* rises if and only if $\varepsilon_{r_2,\eta}^* > \varepsilon_{r_1,\eta}^*$.

To understand the elasticities $\varepsilon_{w_i,\eta}^*$ and $\varepsilon_{r_i,\eta}^*$, note that a shift in $\bar{H}/\bar{L}$ impacts wages and salaries through two channels. First, there is the direct effect described in part (iii) of Lemma 1, which adds $1-\gamma_i$ to $\varepsilon_{w_i,\eta}^*$ and $-\gamma_i$ to $\varepsilon_{r_i,\eta}^*$. This stems from the fact that, with constant boundaries, factor endowments do not affect the matching functions. But given factor intensity differences across sectors, equations (47) and (48) imply that with no changes in matching the right-hand side of each one of these equations equals $\gamma_1 - \gamma_2$, which generate an increase in q_L^* and q_H^* if and only if $\gamma_1 - \gamma_2 > 0$. These shifts in the cutoffs trigger re-matching in each sector, which impacts in turn the wage and salary functions, as implied by Lemmas 3-6 and Corollaries 1 and 2 to Lemma 6. In other words, the impact effect of a rise in $\bar{H}/\bar{L}$ increases the cutoffs for both workers and managers, but we also have to account for the induced change in matching in order to obtain the full effect. To this end, we now express the elasticities $\varepsilon_{w_i,\eta}^*$ and $\varepsilon_{r_i,\eta}^*$ as follows:

$$\varepsilon_{w_i,\eta}^* = (1-\gamma_i)\,\hat{\eta} + \varepsilon_{w_iL}^* \hat{q}_L^* + \varepsilon_{w_iH}^* \hat{q}_H^*, \quad i = 1,2, \tag{49}$$

$$\varepsilon_{r_i,\eta}^* = -\gamma_i \hat{\eta} + \varepsilon_{r_iL}^* \hat{q}_L^* + \varepsilon_{r_iH}^* \hat{q}_H^*, \quad i = 1,2, \tag{50}$$

where $1-\gamma_i$ and $-\gamma_i$ represent the direct impacts of $\bar{H}/\bar{L}$, $\varepsilon_{w_iL}^*$ is the elasticity of $w_i(\cdot)$ with respect to the boundary q_L^* through the induced re-matching (evaluated at q_L^*), and $\varepsilon_{w_iH}^*$ is the elasticity of $w_i(\cdot)$ with respect to the boundary q_H^* through the induced re-matching (evaluated at q_L^*). From (35) and (36) we also have

$$\varepsilon_{r_iF}^* = -\frac{\gamma_i}{1-\gamma_i}\varepsilon_{w_iF}^*, \quad F = H, L; \quad i = 1,2. \tag{51}$$

Now substitute these equations into (47) and (48) to obtain

$$M_h^{HH/LL} \begin{pmatrix} \hat{q}_L^* \\ \hat{q}_H^* \end{pmatrix} = \begin{pmatrix} \gamma_1 - \gamma_2 \\ \gamma_1 - \gamma_2 \end{pmatrix} \hat{\eta}, \tag{52}$$

where

$$M_h^{HH/LL} = \begin{pmatrix} q_L^* \left[\frac{w_1'(q_L^*)}{w_1(q_L^*)} - \frac{w_2'(q_L^*)}{w_2(q_L^*)}\right] + \varepsilon_{w_1L}^* - \varepsilon_{w_2L}^* & \varepsilon_{w_1H}^* - \varepsilon_{w_2H}^* \\ \frac{\gamma_2 \varepsilon_{w_2L}^*}{1-\gamma_2} - \frac{\gamma_1 \varepsilon_{w_1L}^*}{1-\gamma_1} & q_H^* \left[\frac{r_1'(q_H^*)}{r_1(q_H^*)} - \frac{r_2'(q_H^*)}{r_2(q_H^*)}\right] + \frac{\gamma_2 \varepsilon_{w_2H}^*}{1-\gamma_2} - \frac{\gamma_1 \varepsilon_{w_1H}^*}{1-\gamma_1} \end{pmatrix}.$$

From Lemmas 3-6 we have

$$\varepsilon_{w_1L}^* > 0, \quad \varepsilon_{w_2L}^* < 0, \quad \varepsilon_{w_1H}^* < 0, \quad \varepsilon_{w_2H}^* > 0.$$

These equations provide a solution to $\hat{q}_L^*$ and $\hat{q}_H^*$.

The determinant of the matrix $M_h^{HH/LL}$ is

$$\begin{aligned} D_{M_h^{HH/LL}} &= \left\{ q_L^* \left[\frac{w_1'(q_L^*)}{w_1(q_L^*)} - \frac{w_2'(q_L^*)}{w_2(q_L^*)} \right] + \varepsilon_{w_1 L}^* - \varepsilon_{w_2 L}^* \right\} q_H^* \left[\frac{r_1'(q_H^*)}{r_1(q_H^*)} - \frac{r_2'(q_H^*)}{r_2(q_H^*)} \right] \\ &+ \left(\frac{\gamma_2 \varepsilon_{w_2 H}^*}{1-\gamma_2} - \frac{\gamma_1 \varepsilon_{w_1 H}^*}{1-\gamma_1} \right) q_L^* \left[\frac{w_1'(q_L^*)}{w_1(q_L^*)} - \frac{w_2'(q_L^*)}{w_2(q_L^*)} \right] - \frac{\gamma_1 - \gamma_2}{(1-\gamma_1)(1-\gamma_2)} \left(\varepsilon_{w_2 H}^* \varepsilon_{w_1 L}^* - \varepsilon_{w_1 H}^* \varepsilon_{w_2 L}^* \right). \end{aligned}$$

The first two terms on the right-hand side are positive. We now show that the third term also is positive. To this end, note from Lemma 2 that if we change a single boundary and the new boundary is on the original matching function then the new matching function coincides with the old one in the overlapping range of abilities. Therefore, if we choose $dq_L^* = m_i'(q_H^*)\, dq_H^*$, where $m_i(\cdot)$ is the solution of matching in sector i, then a change in the boundary (dq_H^*, dq_L^*) does not change the wage $w_i(q_L^*)$. In other words,

$$\varepsilon_{w_i H}^* + \varepsilon_{w_i L}^* \varepsilon_{m_i}^* = 0,$$

where $\varepsilon_{m_i}^*$ is the elasticity of $m_i(\cdot)$ evaluated at q_H^*. On the other hand, (37) implies for the HH/LL case that

$$\varepsilon_{m_i}^* = \frac{\kappa_m \gamma_i}{1-\gamma_i},$$

where

$$\kappa_m = \frac{\bar{H} r(q_H^*)\, \phi_H(q_H^*)\, q_H^*}{\bar{L} w(q_L^*)\, \phi_L(q_L^*)\, q_L^*}.$$

Therefore,

$$\varepsilon_{w_i H}^* = -\frac{\kappa_m \gamma_i}{1-\gamma_i} \varepsilon_{w_i L}^*.$$

Using this expression, we obtain

$$-\frac{\gamma_1 - \gamma_2}{(1-\gamma_1)(1-\gamma_2)} \left(\varepsilon_{w_2 H}^* \varepsilon_{w_1 L}^* - \varepsilon_{w_1 H}^* \varepsilon_{w_2 L}^* \right) = -\frac{(\gamma_1 - \gamma_2)^2 \kappa_m \varepsilon_{w_1 L}^* \varepsilon_{w_2 L}^*}{(1-\gamma_1)^2 (1-\gamma_2)^2} > 0,$$

which proves that $D_{M_h^{HH/LL}} > 0$.

Solving (52) implies that $\hat{q}_L^* > 0$ and $\hat{q}_H^* > 0$ if and only if $(\gamma_1 - \gamma_2)\,\hat{\eta} > 0$. In other words, a rise in $\bar{H}/\bar{L}$ increases both cutoffs if and only if sector 1 is labor intensive.

Next consider the effects of price changes. An increase in the price of good i raises on impact wages and salaries in sector i by $\hat{p}_i$ and has no direct impact on wages and salaries in the other sector. Following the previous arguments, the change in the equilibrium cutoff points can be found as the solution to

$$M_h^{HH/LL} \begin{pmatrix} \hat{q}_L^* \\ \hat{q}_H^* \end{pmatrix} = \begin{pmatrix} \hat{p}_2 - \hat{p}_1 \\ \hat{p}_2 - \hat{p}_1 \end{pmatrix}, \tag{53}$$

where the matrix $M_h^{HH/LL}$ is the same as in (52). It follows from this system that $\hat{q}_L^* > 0$ and $\hat{q}_H^* > 0$ if and only if $\hat{p}_2 > \hat{p}_1$. That is, an increase in the relative price of good 2 raises both cutoffs and therefore raises output in sector 2 and reduces that in sector 1.

HL/LH Equilibrium

In an HL/LH equilibrium, the cutoffs $\{q_H^*, q_L^*\}$ also satisfy the continuity conditions (43) and (44), but the boundary conditions are different. Assuming as before that the best workers sort into sector 1, this means that in an HL/LH equilibrium the best managers sort into sector 2 and the boundary conditions are

$$m_1(q_{H\min}) = q_L^*, \quad m_1(q_H^*) = q_{L\max},$$

$$m_2(q_H^*) = q_{L\min}, \quad m_2(q_{H\max}) = q_L^*.$$

Figure 5 depicts the pattern of sorting and matching in this type of equilibrium. The more-able workers sort into sector 1 only if

$$\frac{w_1'(q_L^*)}{w_1(q_L^*)} > \frac{w_2'(q_L^*)}{w_2(q_L^*)}$$

and the more-able managers sort into sector 2 only if

$$\frac{r_1'(q_H^*)}{r_1(q_H^*)} < \frac{r_2'(q_H^*)}{r_2(q_H^*)}.$$

To derive the comparative statics, we use as before conditions (47) and (48), which apply in this case too. We also can use the decomposition of elasticities (49) and (50), which still apply. Now, however, the relationship between the elasticities of the salary and wage functions, as described by (51), does not apply, because workers of ability q_L^* do not pair with managers of ability q_H^*, as is evident from Figure 5. Instead, from (35) and (36) we now obtain

$$\varepsilon_{r_1F}^* = -\frac{\gamma_1}{1-\gamma_1}\varepsilon_{w_1F}^{\max}, \quad F = H, L,$$

$$\varepsilon_{r_2F}^* = -\frac{\gamma_2}{1-\gamma_2}\varepsilon_{w_2F}^{\min}, \quad F = H, L,$$

where $\varepsilon_{r_iF}^*$ is defined in the same way as before, $\varepsilon_{w_1F}^{\max}$ is the elasticity of $w_1(\cdot)$ with respect to the boundary q_F^* through the induced re-matching in sector 1 (evaluated at $q_{L\max}$) and $\varepsilon_{w_2F}^{\min}$ is the elasticity of $w_2(\cdot)$ with respect to the boundary q_F^* through the induced re-matching in sector 2 (evaluated at $q_{L\min}$). Using these results the systems of equations (52) and (53) are replaced by

$$M_h^{HL/LH}\begin{pmatrix}\hat{q}_L^* \\ \hat{q}_H^*\end{pmatrix} = \begin{pmatrix}\gamma_1-\gamma_2 \\ \gamma_1-\gamma_2\end{pmatrix}\hat{\eta}, \tag{54}$$

and

$$M_h^{HL/LH}\begin{pmatrix}\hat{q}_L^* \\ \hat{q}_H^*\end{pmatrix} = \begin{pmatrix}\hat{p}_2-\hat{p}_1 \\ \hat{p}_2-\hat{p}_1\end{pmatrix}, \tag{55}$$

where

$$M_h^{HL/LH} = \begin{pmatrix} q_L^*\left[\frac{w_1'(q_L^*)}{w_1(q_L^*)} - \frac{w_2'(q_L^*)}{w_2(q_L^*)}\right] + \varepsilon_{w_1L}^* - \varepsilon_{w_2L}^* & \varepsilon_{w_1H}^* - \varepsilon_{w_2H}^* \\ \frac{\gamma_2\varepsilon_{w_2L}^{\min}}{1-\gamma_2} - \frac{\gamma_1\varepsilon_{w_1L}^{\max}}{1-\gamma_1} & q_H^*\left[\frac{r_1'(q_H^*)}{r_1(q_H^*)} - \frac{r_2'(q_H^*)}{r_2(q_H^*)}\right] + \frac{\gamma_2\varepsilon_{w_2H}^{\min}}{1-\gamma_2} - \frac{\gamma_1\varepsilon_{w_1H}^{\max}}{1-\gamma_1}\end{pmatrix}. \tag{56}$$

From Lemmas 3-6, we have $\varepsilon_{w_1L}^* > 0 > \varepsilon_{w_2L}^*$, $\varepsilon_{w_1H}^* > 0 > \varepsilon_{w_2H}^*$, $\varepsilon_{r_1H}^* < 0 < \varepsilon_{r_2H}^*$, $\varepsilon_{r_1L}^* < 0 < \varepsilon_{r_2L}^*$. This implies that both entries in the top row in (56) are strictly positive and both entries in the bottom row are strictly negative.

Consider (55). The previous observations imply that a positive term $\hat{p}_2 - \hat{p}_1$ either raises q_L^* and reduces q_H^*, or it reduces q_L^* and raises q_H^*. The cutoffs cannot both move in the same direction, because the effect in the top row on the left hand side of (55) would then be opposite to those in the bottom row, whereas on the right hand side both effects have the same sign. We will show that only a rise in q_L^* and a reduction q_H^* can be associated with equilibrium responses, which implies that the determinant of $M_h^{HL/LH}$ must be negative ($D_{M_h^{HL/LH}} < 0$). To prove this, consider an increase in the price p_2 to $p_2' > p_2$ while the price p_1 stays constant. Let X_1 and X_2 denote the output in each sector prior to the price change, and let X_1' and X_2' denote the corresponding output after the price change. Since only prices have changed (and not endowments), under each set of prices both the outputs (X_1, X_2) and (X_1', X_2') are feasible. Since the competitive equilibrium is efficient, the value of output is maximized given prices, which implies that

$$\begin{aligned} p_1 X_1 + p_2 X_2 &\geq p_1 X_1' + p_2 X_2', \\ p_1 X_1 + p_2' X_2 &\leq p_1 X_1' + p_2' X_2', \end{aligned}$$

where the first inequality states that prior to the price change the value of output is higher under production bundle (X_1, X_2) than under (X_1', X_2'), while the opposite holds after the price change. Subtracting and rearranging gives

$$(p_2 - p_2')(X_2 - X_2') \geq 0,$$

which implies that $X_2 \leq X_2'$. An increase in output in sector two cannot be achieved with a fall in q_L^* and a rise q_H^*, because in this case there would be less worker types and less manager types in sector 2. Therefore, an increase in the relative price of good 2 leads to a rise in q_L^* and a reduction q_H^*. This requires $D_{M_h^{HL/LH}} < 0$.

Now consider system (54). Since $D_{M_h^{HL/LH}} < 0$, a rise in the relative endowment $\eta \equiv \bar{H}/\bar{L}$ of managers raises q_L^* and reduces q_H^*. Finally, we must determine the effect of a change in the relative endowment of managers on relative outputs, which is affected both by re-matching and the change in endowments. Sector i pays managers a fraction $1 - \gamma_i$ of revenue. Therefore, in an HL/LH equilibrium, we have

$$(1 - \gamma_1)\, p_1 X_1 = \bar{H} \int_{q_{H\min}}^{q_H^*} r\,(q_H)\, \phi_H\,(q_H)\, dq_H,$$

$$(1 - \gamma_2)\, p_2 X_2 \equiv \bar{H} \int_{q_H^*}^{q_{H\max}} r\,(q_H)\, \phi_H\,(q_H)\, dq_H,$$

which implies

$$\frac{(1 - \gamma_2)\, p_2 X_2}{(1 - \gamma_1)\, p_1 X_1} = \frac{\int_{q_H^*}^{q_{H\max}} r\,(q_H)\, \phi_H\,(q_H)\, dq_H}{\int_{q_{H\min}}^{q_H^*} r\,(q_H)\, \phi_H\,(q_H)\, dq_H}.$$

From (18) we obtain

$$\ln r_i\,(q_H) - \ln r_i\,(q_{H0}) = \int_{q_{H0}}^{q_H} \frac{\psi_{iH}\,[x, m\,(x)]}{(1 - \gamma_i)\, \psi_i\,[x, m\,(x)]} dx, \text{ for all } q_H, q_{H0} \in Q_{Hi}.$$

Substituting this equation into the previous one yields

$$\frac{X_2}{X_1}=\frac{(1-\gamma_1)\,p_1\int_{q_H^*}^{q_{H\max}}\exp\left[\int_{q_H^*}^{q_H}\frac{\psi_{2H}[q,m(q)]}{(1-\gamma_2)\psi_2[q,m(q)]}dq\right]\phi_H\left(q_H\right)dq_H}{(1-\gamma_2)\,p_2\int_{q_{H\min}}^{q_H^*}\exp\left[-\int_{q_H}^{q_H^*}\frac{\psi_{1H}[q,m(q)]}{(1-\gamma_1)\psi_1[q,m(q)]}dq\right]\phi_H\left(q_H\right)dq_H}, \tag{57}$$

where we have used the property that $r(\cdot)$ is a continuous function. When sector 2 is manager intensive, q_H^* is lower in country A, which has more managers per worker. We have shown above that, in such circumstances, managers of a given type are teamed with higher-ability workers in country A. Due to the strict log supermodularity of the productivity functions this implies that $\psi_{iH}\left[q,m\left(q\right)\right]/\psi_i\left[q,m\left(q\right)\right]$ is higher in country A in both sectors. It follows that the impact of a higher $\bar{H}/\bar{L}$ on matching raises the relative output of good 2. In the opposite case, when $\gamma_1<\gamma_2$, the shift in matching reduces the relative output of good 2. In short, the shift in matching raises the relative output of the manager-intensive good.

To complete the analysis of the impact of factor endowments on relative outputs, we need to assess the direct impact of the cutoff q_H^* on the relative outputs in (57). First note that q_H^* affects relative outputs through the boundaries of four integrals. When $\gamma_1>\gamma_2$ and q_H^* declines in response to an increase in $\bar{H}/\bar{L}$, the shifts in the boundaries of the outer integrals in the numerator and denominator raise the relative output of good 2. In the opposite case, when $\gamma_1<\gamma_2$ and q_H^* rises, the relative output of good 2 declines. A shift in the boundaries of the two inner integrals in the numerator and denominator have opposite effects from one another. Consequently, we need to evaluate their relative strength. Differentiation with respect to these boundaries yields:

$$-\frac{X_2}{X_1}\left\{\lim_{q\searrow q_H^*}\frac{\psi_{2H}\left[q,m\left(q\right)\right]}{(1-\gamma_2)\,\psi_2\left[q,m\left(q\right)\right]}-\lim_{q\nearrow q_H^*}\frac{\psi_{1H}\left[q,m\left(q\right)\right]}{(1-\gamma_1)\,\psi_1\left[q,m\left(q\right)\right]}\right\}.$$

Since the best managers sort into sector 2, this requires the slope of the salary function $r(\cdot)$ to be steeper at q_H^* in sector 2, or, using (18), it requires the term in the curly bracket to be positive. It follows that a decline in q_H^* raises the relative output of good 2. If instead good 2 is labor intensive, q_H^* rises in response to an increase in $\bar{H}/\bar{L}$, which raises the relative output of good 1. In either case, country A produces relatively more of the manager-intensive good.

Appendix B

In this appendix, we analyze the limiting case of Cobb-Douglas productivity; that is

$$\psi_i\left(q_H, q_L\right)= q_H^{\beta_i} q_L^{\alpha_i} \text{ for } i=1,2;\ \alpha_i, \beta_i > 0. \tag{58}$$

Note that, in this case, productivity is a weakly log supermodular function of the two ability levels. As such, the complementarity between the talent of workers and that of the manager is somewhat muted compared to what arises with strict log supermodularity, which means that the forces for positive assortative matching within a sector are correspondingly weaker.

There is no need to go through all the steps of a firm's profit maximization problem, because the derivation proceeds much as for the case with homogeneous managers in Section 3. Suffice it to say that the demand per manager for workers of ability q_L by a firm in industry i that pairs these workers with a manager of ability q_H is given by

$$\ell\left(q_L, q_H\right)= \left[\frac{\gamma_i p_i q_H^{\beta_i} q_L^{\alpha_i}}{w\left(q_L\right)}\right]^{\frac{1}{1-\gamma_i}}. \tag{59}$$

Substituting (59) into the expression for profits yields

$$\tilde{\pi}_i\left(q_L, q_H\right)= \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \left(q_H^{\beta_i} q_L^{\alpha_i}\right)^{\frac{1}{1-\gamma_i}} w\left(q_L\right)^{-\frac{\gamma_i}{1-\gamma_i}} - r\left(q_H\right), \tag{60}$$

where $r(q_H)$ is the salary of a manager with ability q_H and $\bar{\gamma}_i \equiv \gamma_i^{\frac{\gamma_i}{1-\gamma_i}}\left(1-\gamma_i\right)$. Every firm chooses the ability of its workers and the ability of its manager so as to maximize profits, yet free entry dictates that these profits must be equal to zero in equilibrium. Let M_i be the set of all matches that maximize profits in sector i. For each pairing (q_L, q_H) in M_i,

$$r\left(q_H\right)= \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \left(q_H^{\beta_i} q_L^{\alpha_i}\right)^{\frac{1}{1-\gamma_i}} w\left(q_L\right)^{-\frac{\gamma_i}{1-\gamma_i}}, \quad i=1,2, \tag{61}$$

by dint of the zero-profit condition. Profit maximization with respect to the choice of types, evaluated for pairings that achieve zero profits in accordance with (61), yields the first-order conditions,

$$\frac{\alpha_i}{\gamma_i}= \varepsilon_w(q_L) \quad \text{for } q_L \in Q_{Li}^{int} \tag{62}$$

and

$$\frac{\beta_i}{1-\gamma_i}= \varepsilon_r(q_H) \quad \text{for } q_H \in Q_{Hi}^{int}. \tag{63}$$

Equation (62) is the analog to (4) and equates the ratio of the elasticities of output with respect to worker ability and labor quantity to the elasticity of the wage schedule. Equation (63) has a similar interpretation regarding a firm's choice of manager type.

In equilibrium, all worker types must be employed, which means that firms in some sector (or both) must demand the full range of workers. Equation (62) can be satisfied for a range of workers only if the wage schedule has a constant elasticity over this range. Therefore, the equilibrium wage schedule must take the form

$$w\left(q_L\right)=w_i q_L^{\alpha_i/\gamma_i} \text{ for } q_L\in Q_{Li}^{int}. \tag{64}$$

The salary schedule for managers must have a similar form, namely

$$r\left(q_H\right)=r_i q_H^{\beta_i/(1-\gamma_i)} \text{ for } q_H\in Q_{Hi}^{int}, \tag{65}$$

where r_i is a "salary anchor" analogous to w_i.

When the wage function has a constant elasticity equal to α_i/γ_i for a range of worker types, a firm in sector i is indifferent as to its choice of employees among workers in this range, irrespective of the ability of its manager. And when the salary function has an elasticity equal to $\beta_i/\left(1-\gamma_i\right)$, the firm is indifferent to the ability of its managers. Accordingly, the matching of workers and managers among those that sort to sector i is indeterminate in the Cobb-Douglas case. This indeterminacy reflects the fact that the productivity function in (58) is only weakly log supermodular and thus provides no clear incentives for positive (or negative) assortative matching.

Although the matching of workers and managers in a sector is not determined in the Cobb-Douglas case, the sorting of these factors to the two sectors follows a familiar pattern. The elasticity of the wage schedule must be greater along its upper segment than along its lower segment, or else firms that hire the less able workers would all prefer to upgrade their employees. Similarly, the elasticity of the salary schedule must be greater along its upper segment than its lower segment. We designate as sector 1 whichever industry has the greater ratio of the output elasticity with respect to worker ability to the output elasticity with respect to labor quantity. With this labeling convention, $s_L=\alpha_1/\gamma_1-\alpha_2/\gamma_2>0$. Then, in any equilibrium in which a country produces both goods, sector 1 attracts the workers with ability q_L above some cutoff q_L^*. If $s_H=\beta_1/\left(1-\gamma_1\right)-\beta_2/\left(1-\gamma_2\right)>0$, then sector 1 also attracts the more able managers with $q_H>q_H^*$; otherwise, the sorting of managers is opposite to that for workers.

For precision, we state more formally the environment we consider throughout this appendix and the sorting pattern that results.[41]

Assumption 3 (i) $S_H=\left[q_{H\min},q_{H\max}\right]$, $0<q_{H\min}<q_{H\max}<+\infty$; (ii) $\psi_i\left(q_H,q_L\right)=q_H^{\beta_i}q_L^{\alpha_i}$, $\alpha_i,\beta_i>0$, for $i=1,2$; and (iii) $s_L\equiv\alpha_1/\gamma_1-\alpha_2/\gamma_2>0$.

Proposition 10 *Suppose that Assumption 3 holds. Then, in any competitive equilibrium with employment in both sectors, the more able workers with $q_L\geq q_L^*$ are employed in sector 1 and the less able workers with $q_L\leq q_L^*$ are employed in sector 2, for some $q_L^*\in S_L$. If $s_H>0$ $(s_H<0)$, the more able managers with $q_H\geq q_H^*$ are employed in sector 1 (sector 2) and the less able managers with $q_H\leq q_H^*$ are employed in sector 2 (sector 1), for some $q_H^*\in S_H$.*

To describe the equilibrium once the sorting pattern has been settled, we invoke factor-market clearing, continuity of worker wages, continuity of managerial salaries, and the zero-profit conditions. For concreteness,

[41] Proofs of all Propositions stated in this appendix are provided at the end.

let us focus on the case in which $s_H > 0$ so that the more able managers sort to industry 1; the opposite case can be handled similarly.

It proves convenient to define $e_{Hi}(q_H) = q_H^{\beta_i/(1-\gamma_i)}$ as the effective managerial input of a manager with ability q_H who works in sector i. Then the aggregate supplies of effective managerial input in sectors 1 and 2 are

$$H_1 = \bar{H} \int_{q_H^*}^{q_{H\max}} q_H^{\frac{\beta_1}{1-\gamma_1}} \phi_H(q_H)\, dq_H, \tag{66}$$

and

$$H_2 = \bar{H} \int_{q_{H\min}}^{q_H^*} q_H^{\frac{\beta_2}{1-\gamma_2}} \phi_H(q_H)\, dq_H\ , \tag{67}$$

respectively. Note that $H_1/\bar{H}$ depends only on q_H^* and is a monotonically decreasing function, and $H_2/\bar{H}$ also depends only on q_H^* and is monotonically increasing.

Consider now the supply and demand for effective labor in sector 1, where we define $e_{Li}(q_L) = q_L^{\alpha_i/\gamma_i}$ as the effective labor provided by a worker of ability q_L in sector i. From the labor demand equation (59), a firm in sector 1 combines a manager with e_{Hi} units of effective managerial input with $e_{Hi}(\gamma_i p_i/w_i)^{1/(1-\gamma_i)}$ units of effective labor. Therefore, the H_1 units of effective managerial input that are hired into sector 1 are combined with $H_1(\gamma_1 p_1/w_1)^{1/(1-\gamma_1)}$ units of effective labor. Noting the definition of H_1 and equating the demand for effective labor in sector 1 with the supply of effective labor among those with ability above q_L^*, we have

$$\bar{H}\left(\frac{\gamma_1 p_1}{w_1}\right)^{\frac{1}{1-\gamma_1}} \int_{q_H^*}^{q_{H\max}} q_H^{\frac{\beta_1}{1-\gamma_1}} \phi_H(q_H)\, dq_H = \bar{L} \int_{q_L^*}^{q_{L\max}} q_L^{\frac{\alpha_1}{\gamma_1}} \phi_L dq_L\ . \tag{68}$$

A similar condition applies in sector 2, where labor-market clearing requires

$$\bar{H}\left(\frac{\gamma_2 p_2}{w_2}\right)^{\frac{1}{1-\gamma_2}} \int_{q_{H\min}}^{q_H^*} q_H^{\frac{\beta_2}{1-\gamma_2}} \phi_H(q_H)\, dq_H = \bar{L} \int_{q_{L\min}}^{q_L^*} q_L^{\frac{\alpha_2}{\gamma_2}} \phi_L dq_L\ . \tag{69}$$

Continuity of the wage schedule at q_L^* requires that

$$w_1 (q_L^*)^{\frac{\alpha_1}{\gamma_1}} = w_2 (q_L^*)^{\frac{\alpha_2}{\gamma_2}}\ . \tag{70}$$

The salary function for managers must also be continuous and firms that hire managers with ability q_H^* must earn zero profits in either sector. Together, these considerations imply

$$\bar{\gamma}_1 p_1^{\frac{1}{1-\gamma_1}} w_1^{-\frac{\gamma_1}{1-\gamma_1}} (q_H^*)^{\frac{\beta_1}{1-\gamma_1}} = \bar{\gamma}_2 p_2^{\frac{1}{1-\gamma_2}} w_2^{-\frac{\gamma_2}{1-\gamma_2}} (q_H^*)^{\frac{\beta_2}{1-\gamma_2}}\ . \tag{71}$$

Equations (68)-(71) comprise four equations that can be used to solve for the two wage anchors, w_1 and w_2, and the two cutoffs, q_L^* and q_H^*. The effective supply of managers in sectors 1 and 2, H_1 and H_2, can then be solved from (66) and (67). Finally, the salary anchors for the managers can be computed from the zero-profit conditions, which imply

$$r_i = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} w_i^{-\frac{\gamma_i}{1-\gamma_i}} \quad \text{for } i = 1, 2. \tag{72}$$

This completes our characterization of the supply-side equilibrium for an economy that faces prices p_1 and p_2.

Pattern of Trade

As in Section 3, we need an expression for an economy's relative outputs in order to conduct the comparative static analysis that reveals the pattern of trade between countries that differ in their relative factor endowments or in their distributions of factor types. The H_i units of effective managers employed in sector i collectively produce $X_i = H_i (\gamma_i p_i)^{\gamma_i/(1-\gamma_i)} w_i^{-\gamma_i/(1-\gamma_i)}$ units of good i. Each effective unit of managerial input is paid a salary of r_i in sector i and—by continuity of the salary function—$r_1/r_2 = (q_H^*)^{-s_H}$ (see (65)). Using this condition together with (68)-(69) and (71)-(72), we can write

$$\frac{X_1}{X_2} = \frac{r_1 H_1}{r_2 H_2} \frac{(1-\gamma_2) p_2}{(1-\gamma_1) p_1} = \frac{(1-\gamma_2) p_2 \int_{q_H^*}^{q_{H\max}} q_H^{\frac{\beta_1}{1-\gamma_1}} \phi_H(q_H) \, dq_H}{(1-\gamma_1) p_1 \int_{q_{H\min}}^{q_H^*} q_H^{\frac{\beta_2}{1-\gamma_2}} \phi_H(q_H) \, dq_H} (q_H^*)^{-s_H} . \tag{73}$$

Similar to the case of homogeneous managers, the first equality reflects the fact that the aggregate salaries of all managers in sector i absorb a fraction $1-\gamma_i$ of revenue. And the second equality implies that, since $s_H > 0$ in the case under consideration, X_1/X_2 is a decreasing function of q_H^*. Therefore, to identify the pattern of trade, we need only find which country allocates more effective managerial input to sector 1 relative to its aggregate endowment of managers; that is, how q_H^* varies with factor endowments.[42]

The system of equations (68)-(71) that applies with Cobb-Douglas productivity is quite similar to the system (6)-(9) that applies when managers are homogeneous, except that now we need to use the effective managerial input in a sector in place of the pure number of managers. In other words, the multiplicative separability of the productivity function allows us to construct an aggregate measure of managerial input that plays the same role as does the number of managers when managers are equally productive. We can do so, because there are no forces present in the Cobb-Douglas case to induce any particular pattern of matching within either sector. The following propositions assert that the determinants of the trade pattern in an economy with heterogeneous managers but Cobb-Douglas productivity mirror those that we described for an economy with homogeneous managers.

Proposition 11 *Suppose that Assumption 3 holds. Then if $\phi_L^A(q_L) = \phi_L^B(q_L)$ for all $q_L \in S_L^A = S_L^B$, $\phi_H^A(q_H) = \phi_H^B(q_H)$ for all $q_H \in S_H^A = S_H^B$, and $\bar{H}^A/\bar{L}^A > \bar{H}^B/\bar{L}^B$, country A exports the manager-intensive good.*

Proposition 12 *Suppose that Assumption 3 holds and $\bar{H}^A/\bar{L}^A = \bar{H}^B/\bar{L}^B$. Then, (i) if $\phi_H^A(q_H) = \phi_H^B(q_H)$ for all $q_H \in S_H^A = S_H^B$ and $\phi_L^A(q_L)$ is a rightward shift of $\phi_L^B(q_L)$ for some $\lambda > 1$, then country A exports good 1 if and only if $\alpha_1 > \alpha_2$; (ii) if $\phi_L^A(q_L) = \phi_L^B(q_L)$ for all $q_L \in S_L^A = S_L^B$ and $\phi_H^A(q_H)$ is a rightward shift of $\phi_H^B(q_H)$ for some $\lambda > 1$, then country A exports good 1 if and only if $\beta_1 > \beta_2$.*

In short, the Heckscher-Ohlin theorem applies when countries have similar distributions of factor types but differ in their relative aggregate endowments of managers versus workers. Alternatively, if the relative factor

[42] Note that in the opposite case, when $s_H < 0$, managers with $q_H \geq q_H^*$ sort into sector 2 while managers with $q_H \leq q_H^*$ sort into sector 1. As a result, X_1/X_2 is an increasing function of q_H^*.

endowments are the same in the two countries but they differ in their distributions of one of the factors, then the country with the rightward-shifted distribution of a factor exports the good produced by the industry in which productivity responds more elastically to that factor's ability.

Effects of Trade on Income Distribution

Our results on income distribution also carry over straightforwardly from the case with homogeneous managers to that with manager heterogeneity but Cobb-Douglas productivity. First note that within-industry income distribution is not affected by world trade inasmuch as the elasticity of the wage schedule for workers employed in a given industry is constant. As a result, (64) implies that $w\left(q_L'\right)/w\left(q_L''\right) = \left(q_L'/q_L''\right)^{\alpha_i/\gamma_i}$ for $q_L', q_L'' \in Q_{Li}$ and (65) implies that $r\left(q_H'\right)/r\left(q_H''\right) = \left(q_H'/q_H''\right)^{\beta_i/(1-\gamma_i)}$ for $q_H', q_H'' \in Q_{Hi}$. Second, relative rewards of workers and managers that are employed in different industries do change with trade, inasmuch as the wage and salary anchors w_i and r_i change. Our next proposition is

Proposition 13 *Suppose that Assumption 3 holds and $s_H \approx 0$. When $\hat{p}_1 > 0$, (i) $\hat{w}_1 > \hat{w}_2$; (ii) if $\gamma_1 \approx \gamma_2$, then $\hat{w}_1 > \hat{p}_1 > \hat{r}_1 \approx \hat{r}_2 > 0 > \hat{w}_2$; (iii) if $\gamma_1 > \gamma_2$ and $s_L \approx 0$, then $\hat{w}_1 \approx \hat{w}_2 > \hat{p}_1 > 0 > \hat{r}_1 \approx \hat{r}_2$; (iv) if $\gamma_1 < \gamma_2$ and $s_L \approx 0$, then $\hat{r}_1 \approx \hat{r}_2 > \hat{p}_1 > 0 > \hat{w}_1 \approx \hat{w}_2$.*

Proposition 13 can be understood by recognizing that the model with heterogeneous workers and managers also contains a blend of Stolper-Samuelson and Ricardo-Viner forces. When $s_H \approx 0$, there is no difference in the suitability of the various managers for employment in one sector versus the other, because the comparative advantage associated with greater ability of the input just offsets the comparative advantage associated with greater quantity. Then, it is as if managers are a perfectly mobile, homogeneous factor. When s_L also is small, the Stolper-Samuelson forces will dominate, and workers in both industries will see a gain in real income if the relative price of the labor-intensive good rises and will see a loss in real income if the relative price of the labor-intensive good falls. In contrast, if factor intensities are approximately the same in the two industries, the Stolper-Samuelson forces will be muted, and the partial specificity of workers arising from the comparative advantage of ability in sector 1 will govern the income responses. Then, workers will benefit in real terms when the relative price of the good they produce rises and will lose in real terms if the relative price of this good falls. Also note that similar considerations imply that if $s_H > 0$ but $s_L \approx 0$ and $\gamma_1 \approx \gamma_2$, the economy behaves like one with sector-specific managers and perfectly mobile labor. Then $\hat{r}_1 > \hat{p}_1 > \hat{w}_1 \approx \hat{w}_2 > 0 > \hat{r}_2$, i.e., managers in the expanding sector gain, managers in the contracting sector lose, and workers may gain or lose in real terms depending on their consumption pattern. Finally, similarly to Proposition 4, an increase in the price of good 1 raises overall wage inequality, because it does not change relative wages within sectors and it increases wages of the more able, better-paid workers employed in sector 1 relative to the less able, lower-paid workers in sector 2.

Proofs

First note that, in the system comprising (68)-(71), a proportional increase in the number of managers and workers has no effect on the wage anchors w_1 and w_2 or on the ability cutoffs q_L^* and q_H^*. Therefore, it does not change the output ratio X_1/X_2 (see (73)). It follows that if countries A and B differ only in size, with $\bar{H}$ and $\bar{L}$ being proportionately larger in one of the countries, they will have the same relative demand for the two goods and the same relative supply and they will not trade with one another. Accordingly, we can find the impact of $\bar{H}/\bar{L}$ on the pattern of trade by analyzing the impact of $\bar{L}$ on q_H^*, which will tell us how the relative supply X_1/X_2 is affected.

Differentiating the equilibrium system (68)-(71), we obtain

$$\begin{pmatrix} 1 & -1 & s_L & 0 \\ -\frac{\gamma_1}{1-\gamma_1} & \frac{\gamma_2}{1-\gamma_2} & 0 & s_H \\ 0 & \frac{E_2}{1-\gamma_2} & \Lambda_2 & -\Theta_2 \\ \frac{E_1}{1-\gamma_1} & 0 & -\Lambda_1 & \Theta_1 \end{pmatrix} \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{q}_L^* \\ \hat{q}_H^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -E_2 \\ -E_1 \end{pmatrix} \widehat{\bar{L}} + \begin{pmatrix} 0 \\ -\frac{1}{1-\gamma_1} \\ 0 \\ \frac{E_1}{1-\gamma_1} \end{pmatrix} \hat{p}_1,$$

where E_i is effective labor in sector i, defined as

$$E_1 = \bar{H} \left(\frac{\gamma_1 p_1}{w_1} \right)^{\frac{1}{1-\gamma_1}} \int_{q_H^*}^{q_{H\max}} q_H^{\frac{\beta_1}{1-\gamma_1}} \phi_H(q_H)\, dq_H,$$

$$E_2 = \bar{H} \left(\frac{\gamma_2 p_2}{w_2} \right)^{\frac{1}{1-\gamma_2}} \int_{q_{H\min}}^{q_H^*} q_H^{\frac{\beta_2}{1-\gamma_2}} \phi_H(q_H)\, dq_H,$$

and

$$\Lambda_1 = \bar{L} \left(q_L^*\right)^{\frac{\alpha_1}{\gamma_1}+1} \phi_L\left(q_L^*\right),$$

$$\Lambda_2 = \bar{L} \left(q_L^*\right)^{\frac{\alpha_2}{\gamma_2}+1} \phi_L\left(q_L^*\right),$$

$$\Theta_1 = \bar{H} \left(\frac{\gamma_1 p_1}{w_1} \right)^{\frac{1}{1-\gamma_1}} \left(q_H^*\right)^{\frac{\beta_1}{1-\gamma_1}+1} \phi_H\left(q_H^*\right),$$

$$\Theta_2 = \bar{H} \left(\frac{\gamma_2 p_2}{w_2} \right)^{\frac{1}{1-\gamma_2}} \left(q_H^*\right)^{\frac{\beta_2}{1-\gamma_2}+1} \phi_H\left(q_H^*\right).$$

The determinant of the matrix on the left-hand side of this system, D_{CD}, satisfies

$$\begin{aligned} (1-\gamma_2)(1-\gamma_1)(-D_{CD}) &= (\Theta_1\Lambda_2 - \Theta_2\Lambda_1)(\gamma_1-\gamma_2) + s_H\left[\Lambda_1 E_2 (1-\gamma_1) + \Lambda_2 E_1 (1-\gamma_2)\right] \\ &\quad + s_L\left(\Theta_1\gamma_1 E_2 + \Theta_2\gamma_2 E_1\right) + E_1 E_2 s_H s_L. \end{aligned}$$

Using the equilibrium conditions (70) and (71), we find that

$$(\Theta_1\Lambda_2 - \Theta_2\Lambda_1)(\gamma_1-\gamma_2) = \Theta_2\Lambda_1 \frac{(\gamma_1-\gamma_2)^2}{\gamma_2(1-\gamma_1)} > 0.$$

Therefore $D_{CD} < 0$. We also compute

$$\hat{q}_H^* D_{CD} = \left(\Lambda_1 E_2 + \Lambda_2 E_1 + E_1 E_2 s_L\right) \frac{\gamma_1-\gamma_2}{(1-\gamma_1)(1-\gamma_2)} \widehat{\bar{L}}.$$

Since $D_{CD} < 0$, an increase in $\bar{L}$ reduces q_H^* if and only if $\gamma_1 > \gamma_2$. So, the output of good 1 rises relative to that of good 2 if and only if sector 2 is more labor intensive than sector 1. This proves Proposition 11.

Next, we calculate the impact of p_1 on the wage anchors:

$$\begin{aligned}\hat{w}_1(1-\gamma_2)(1-\gamma_1)(-D_{CD}) &= (\Theta_1\Lambda_2-\Theta_2\Lambda_1+\Lambda_2E_1s_H)(1-\gamma_2)\hat{p}_1\\ &\quad+[(\Theta_1E_2+\Theta_2\gamma_2E_1)s_L+E_1E_2s_Hs_L]\hat{p}_1,\\ \hat{w}_2(1-\gamma_1)(-D_{CD}) &= (\Theta_1\Lambda_2-\Theta_2\Lambda_1+\Lambda_2E_1s_H-\Theta_2E_1s_L)\hat{p}_1.\end{aligned}$$

Therefore,

$$(\hat{w}_1-\hat{w}_2)(1-\gamma_1)(-D_{CD})=[(\Theta_1E_2+\Theta_2\gamma_2E_1)s_L+E_1E_2s_Hs_L+\Theta_2E_1s_L(1-\gamma_2)]\hat{p}_1.$$

Since $D_{CD}<0$, it follows that an increase in the price of good 1 results in $\hat{w}_1>\hat{w}_2$, which proves part (i) of Proposition 13.

Next consider the case in which $s_H\approx 0$ and $\gamma_1\approx\gamma_2$. In this case,

$$(1-\gamma_2)(1-\gamma_1)(-D_{CD})\approx s_L(\Theta_1\gamma_1E_2+\Theta_2\gamma_2E_1).$$

Then

$$\hat{w}_1\approx\hat{w}_2\approx\frac{\Theta_1E_2+\Theta_2\gamma_2E_1}{\Theta_1\gamma_1E_2+\Theta_2\gamma_2E_1}\hat{p}_1,$$

because $\gamma_1\approx\gamma_2$ implies $\Theta_1\Lambda_2-\Theta_2\Lambda_1\approx 0$. Evidently, in this case, $\hat{w}_1>\hat{p}_1>0>\hat{w}_2$. To complete the proof of part (ii) of Proposition 13, we need to calculate the response of the anchors r_1 and r_2 for the managers' salaries. When p_1 rises, (72) yields $\hat{r}_1=(1-\gamma_1)^{-1}\hat{p}_1-\gamma_1(1-\gamma_1)^{-1}\hat{w}_1$ and $\hat{r}_2=-\gamma_2(1-\gamma_2)^{-1}\hat{w}_2$. In case (ii) of Proposition 13, with $s_H\approx 0$ and $\gamma_1\approx\gamma_2$, these imply

$$\hat{r}_1\approx\hat{r}_2\approx\frac{\Theta_2\gamma_2E_1}{\Theta_1\gamma_1E_2+\Theta_2\gamma_2E_1}\hat{p}_1.$$

It follows that $\hat{p}_1>\hat{r}_1\approx\hat{r}_2>0$. So, part (ii) of the proposition is proved.

We turn now to parts (iii) and (iv) of Proposition 13. The antecedents $s_H\approx 0$ and $s_L\approx 0$ imply

$$\begin{aligned}(1-\gamma_2)(1-\gamma_1)(-D_{CD}) &\approx (\Theta_1\Lambda_2-\Theta_2\Lambda_1)(\gamma_1-\gamma_2),\\ \hat{w}_1(1-\gamma_1)(-D_{CD}) &\approx (\Theta_1\Lambda_2-\Theta_2\Lambda_1)\hat{p}_1,\\ \hat{w}_2(1-\gamma_1)(-D_{CD}) &\approx (\Theta_1\Lambda_2-\Theta_2\Lambda_1)\hat{p}_1.\end{aligned}$$

It follows that

$$\hat{w}_1\approx\hat{w}_2\approx\frac{1-\gamma_2}{\gamma_1-\gamma_2}\hat{p}_1,$$

which implies that $\hat{w}_1\approx\hat{w}_2>\hat{p}_1>0$ for $\gamma_1>\gamma_2$ and $\hat{w}_1\approx\hat{w}_2<0<\hat{p}_1$ for $\gamma_1<\gamma_2$. Moreover, since $\hat{r}_1=(1-\gamma_1)^{-1}\hat{p}_1-\gamma_1(1-\gamma_1)^{-1}\hat{w}_1$ and $\hat{r}_2=-\gamma_2(1-\gamma_2)^{-1}\hat{w}_2$, we have

$$\hat{r}_1\approx\hat{r}_2\approx-\frac{\gamma_2}{\gamma_1-\gamma_2}\hat{p}_1.$$

Evidently, in this case, $\hat{r}_1\approx\hat{r}_2<0<\hat{p}_1$ when $\gamma_1>\gamma_2$ and $\hat{r}_1\approx\hat{r}_2>\hat{p}_1>0$ when $\gamma_1<\gamma_2$. This completes the proof of Proposition 13.

We next consider the impact of a rightward shift of the density function $\phi(q_L)$, as defined in (12).

To perform these comparative statics, we follow the procedure from the previous section; that is, we hold the distribution of types constant but endow a worker of type q_L with λq_L units of ability, and we define $Q_L^* = \lambda q_L^*$. Differentiating the equilibrium system (68)-(71), we now obtain

$$\begin{pmatrix} 1 & -1 & s_L & 0 \\ -\frac{\gamma_1}{1-\gamma_1} & \frac{\gamma_2}{1-\gamma_2} & 0 & s_H \\ 0 & \frac{E_2}{1-\gamma_2} & \Lambda_2 & -\Theta_2 \\ \frac{E_1}{1-\gamma_1} & 0 & -\Lambda_1 & \Theta_1 \end{pmatrix} \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{Q}_L^* \\ \hat{q}_H^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{\alpha_2}{\gamma_2} E_2 + \Lambda_2 \\ -\frac{\alpha_1}{\gamma_1} E_1 - \Lambda_1 \end{pmatrix} \widehat{\lambda}.$$

Using (**??**), it follows that for $s_H > 0$ an increase in λ raises the relative output of good 1 if it reduces q_H^*. However, from the above system of equations we obtain:

$$\hat{q}_H^* (1-\gamma_2)(1-\gamma_1)(-D_{CD}) = -(\Lambda_1 E_2 + \Lambda_2 E_1 + s_L E_1 E_2)(\alpha_1 - \alpha_2)\hat{\lambda}.$$

Therefore a rightward shift of the density function $\phi(q_L)$ raises the relative output of good 1 if and only if $\alpha_1 > \alpha_2$. This proves the first part of Proposition 12. The second part is proved similarly.